

Weighted Sequences In Finite Cyclic Groups*

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Abstract

Let $p > 7$ be a prime, let $G = \mathbf{Z}/p\mathbf{Z}$, and let $S_1 = \prod_{i=1}^p g_i$ and $S_2 = \prod_{i=1}^p h_i$ be two sequences with terms from G . Suppose that the maximum multiplicity of a term from either S_1 or S_2 is at most $(2p + 1)/5$. Then we show that, for each $g \in G$, there exists a permutation σ of $1, 2, \dots, p$ such that $g = \sum_{i=1}^p (g_i \cdot h_{\sigma(i)})$. The question is related to a conjecture of A. Bialostocki concerning weighted subsequence sums and the Erdős-Ginzburg-Ziv Theorem.

1 Introduction

Let G be a finite abelian group (written additively), and let $\mathcal{F}(G)$ denote the free abelian monoid over G with monoid operation written multiplicatively and given by concatenation, i.e., $\mathcal{F}(G)$ consists of all multi-sets over G , and an element $S \in \mathcal{F}(G)$, which we refer to as a *sequence*, is written in the form $S = \prod_{i=1}^k g_i = \prod_{g \in G} g^{v_g(S)}$ with $g_i \in G$, where $v_g(S) \in \mathbb{N}_0$ is the *multiplicity of g in S* and k is the *length of S* , denoted by $|S| = k$. Set

$$h(S) = \max_{g \in G} \{v_g(S)\}.$$

If $h(S) \leq 1$, then we call S a square-free sequence in G , in which case we may also regard S as a subset of G . A sequence T is a *subsequence of S* , which we denote by $T|S$, if $v_g(T) \leq v_g(S)$ for every $g \in G$. By $\sigma(S)$ we denote the sum of all terms in $S = \prod_{i=1}^k g_i$, that is $\sigma(S) = \sum_{i=1}^k g_i$. If G is also a ring with multiplicative operation denoted \cdot , and $S, T \in \mathcal{F}(G)$ with $S = \prod_{i=1}^{k_1} g_i, T = \prod_{i=1}^{k_2} h_i$ and $r = \min\{k_1, k_2\}$, then define

$$S \cdot T := \left\{ \sum_{i=1}^r (g_{\sigma_1(i)} \cdot h_{\sigma_2(i)}) \mid \sigma_i \text{ a permutation of } 1, 2, \dots, k_i, i = 1, 2 \right\}.$$

In 1996, Y. Caro made the following conjecture [3], which can be regarded as a weighted version of the Erdős-Ginzburg-Ziv Theorem [6] (which is the case $W = 1^n$).

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CONJECTURE 1. Let $G = \mathbf{Z}/n\mathbf{Z}$. If $S, W \in \mathcal{F}(G)$ with $|W| = k$, $\sigma(W) = 0$ and $|S| = n + k - 1$, then $0 \in W \cdot S$.

For prime cyclic groups, Conjecture 1 was confirmed by N. Alon, A. Bialostocki and Y. Caro (see [3]). W. Gao and X. Jin showed, in particular, that Conjecture 1 is true if $n = p^2$ for some prime $p \in \mathbb{P}$ (see [7]), and more recently, a complete confirmation of Conjecture 1 was found by D. Grynkiewicz (see [8]).

On the basis of Conjecture 1, A. Bialostocki made the following conjecture [2].

CONJECTURE 2. Let $G = \mathbf{Z}/n\mathbf{Z}$ with n even. If $S_1, S_2 \in \mathcal{F}(G)$ with $|S_1| = |S_2| = n$ and $\sigma(S_1) = \sigma(S_2) = 0$, then $0 \in S_1 \cdot S_2$.

The example

$$S_1 = 0^{n-2}1(-1), S_2 = 012 \cdots (n-1) \quad (1)$$

was given to show Conjecture 2 could not hold for odd n . Additionally, A. Bialostocki confirmed Conjecture 2 for small numbers using a computer.

A related question is to ask what conditions guarantee that $S_1 \cdot S_2 = G$, so that every element of G , including zero, can be represented as a sum of products between the terms of S_1 and S_2 . Of course, if S_1 has only one distinct term, then $|S_1 \cdot S_2| = 1$, so some condition, say either on the multiplicity of terms or on the number of distinct terms, is indeed needed.

In this paper, we show, for a group of prime order $p > 3$, that $h(S_i)$ being small is enough to guarantee $S_1 \cdot S_2 = G$; note that $S_1 \cdot S_2 = G$ implies $0 \in S_1 \cdot S_2$ as a particular consequence. Our main result is the following.

THEOREM 1. Let $p > 3$ be a prime, let $G = \mathbf{Z}/p\mathbf{Z}$, and let $S_1, S_2 \in \mathcal{F}(G)$ with $|S_1| = |S_2| = p$. If $p \neq 7$ and $\max\{h(S_1), h(S_2)\} \leq \frac{2p+1}{5}$, or $p = 7$ and $\max\{h(S_1), h(S_2)\} \leq 2$, then $S_1 \cdot S_2 = G$.

Let $G = \mathbf{Z}/n\mathbf{Z}$. If $n \equiv -1 \pmod{4}$, then the example

$$S_1 = S_2 = 0^{\frac{n-1}{2}} 1^{\frac{n-1}{2}} \left(\frac{n+1}{2} \right) \quad (2)$$

has $\sigma(S_1) = \sigma(S_2) = 0$, $\max\{h(S_1), h(S_2)\} = \frac{n-1}{2}$ and $0 \notin S_1 \cdot S_2$, giving an additional counterexample to the possibility of Conjecture 2 holding for odd order groups. It also shows that the bound $\max\{h(S_1), h(S_2)\} \leq 2$ for $p = 7$ is tight in Theorem 1. The example $S_1 = S_2 = 0^3 1^2$ shows that the bound for $p = 5$ is tight, and the example given in (1) for $n = 3$ shows that the theorem cannot hold for $p = 3$. Finally, letting $x = \lceil \frac{2n+2}{5} \rceil$, the example

$$S_1 = S_2 = 0^x 1^x 2^{n-2x}, \quad (3)$$

for $n > 7$, has $\max\{h(S_1), h(S_2)\} = \lceil \frac{2n+2}{5} \rceil$ and $S_1 \cdot S_2 \subseteq [n - 2x, 4(n - 2x) + x]$, so that $|S_1 \cdot S_2| \leq 3n - 5x + 1 \leq n - 1$, showing that the bound $\frac{2p+1}{5}$ from Theorem 1 is also best possible.

2 The Proof of the Main Result

To prove Theorem 1, we need some preliminaries. Given subsets A and B of an abelian group G , their sumset is the set of all possible pairwise sums:

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

We use \overline{A} to denote the complement of A in G . For a prime order group, we have the following classical inequality [4].

THEOREM 2. (Cauchy-Davenport Theorem) Let p be a prime, if A and B are nonempty subsets of $\mathbf{Z}/p\mathbf{Z}$, then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

The case when equality holds in the Cauchy-Davenport bound was addressed by A. Vosper [9].

THEOREM 3. (Vosper's Theorem) Let p be a prime, and let $A, B \subseteq \mathbf{Z}/p\mathbf{Z}$ with $|A|, |B| \geq 2$. If

$$|A + B| = |A| + |B| - 1 \leq p - 2,$$

then A and B are arithmetic progressions with common difference. If

$$|A + B| = |A| + |B| - 1 = p - 1,$$

then $A = x - \overline{B}$ for some $x \in \mathbf{Z}/p\mathbf{Z}$.

As an immediate corollary of Theorem 3, we have the following.

COROLLARY 1. Let p be a prime, and let $A, B \subseteq \mathbf{Z}/p\mathbf{Z}$ with $|A|, |B| \geq 2$. If

$$|A + B| = |A| + |B| - 1 < p$$

and B is an arithmetic progression with difference d , then A and $A + B$ are also arithmetic progressions with difference d .

We will also need the following basic proposition (an immediate consequence of Lemma 1 in [5]).

PROPOSITION 1. Let p be a prime, and let $A, B \subseteq \mathbf{Z}/p\mathbf{Z}$ be nonempty with $|B| \geq 3$. If

$$|A + B| \leq |A| + |B| < p,$$

and B is an arithmetic progression with difference d , then $A + B$ is also an arithmetic progression with difference d .

The following will be the key lemma used in the proof of Theorem 1.

LEMMA 1. Let $p > 3$ be a prime, let $G = \mathbf{Z}/p\mathbf{Z}$, and let $U, V \in \mathcal{F}(G)$ be square-free with $|U| = |V| = 3$. Then $|U \cdot V| \geq 4$; furthermore, assuming $p > 7$, then equality is only possible if $U \cdot V$ is not an arithmetic progression and either U and V are both arithmetic progressions or else, up to affine transformation, $U = 01x$ and $V = 01y$ with x and y the two distinct roots of $z^2 - z + 1$.

PROOF. By an appropriate pair of affine transformations (maps of the form $z \mapsto \alpha z + \beta$ with $\alpha, \beta \in \mathbf{Z}/p\mathbf{Z}$ and $\alpha \neq 0$), we may w.l.o.g. assume $U = 01x$ and $V = 01y$ with $x, y \notin \{0, 1\}$. By possibly applying the affine transformation $z \mapsto -z + 1$ to U , we may assume $x \neq y$ unless $x = y = \frac{p+1}{2}$, in which case we may instead assume $x \neq y$ by applying the affine transformation $z \mapsto 2z$ to U and the affine transformation $z \mapsto -2z + 1$ to V . Observe that

$$U \cdot V = \{1, x, y, xy, x + y, xy + 1\}. \quad (4)$$

Hence, since $x, y \notin \{0, 1\}$ and $x \neq y$, it follows that $\{1, x, y\}$ is a cardinality 3 subset of $U \cdot V$. Consequently, if $|U \cdot V| < 4$, then $xy, x + y, xy + 1 \in \{1, x, y\}$.

Since $x, y \notin \{0, 1\}$, if $xy \in \{1, x, y\}$, then

$$xy = 1; \quad (5)$$

if $x + y \in \{1, x, y\}$, then

$$x + y = 1; \quad (6)$$

and if $xy + 1 \in \{1, x, y\}$, then w.l.o.g.

$$xy = x - 1. \quad (7)$$

If (5), (6) and (7) all hold, then (5) and (7) imply $x = 2$, whence (6) implies $y = -1$; thus (5) implies $3 = 0$, contradicting that $p > 3$. As a result, we conclude that $|U \cdot V| \geq 4$. To prove the second part of the lemma, we now assume $p > 7$ and $|U \cdot V| = 4$.

First suppose that either U or V is an arithmetic progression, say U . Thus w.l.o.g. $U = 01(-1)$. If V is also an arithmetic progression, then by an appropriate affine transformation we obtain $V = 012$; consequently, $U \cdot V = \{\pm 1, \pm 2\}$, whence $U \cdot V + U \cdot V = \pm\{0, 1, 2, 3, 4\}$, so that $|U \cdot V + U \cdot V| = 9 > |U \cdot V| + |U \cdot V| - 1$ for $p > 7$, which implies $U \cdot V$ is not an arithmetic progression, as desired. Therefore we may instead assume V is not an arithmetic progression.

Since $U = 01(-1)$, $y \neq x$ and $y \notin \{0, 1\}$, it follows in view of (4) that $\{\pm 1, \pm y\}$ is a cardinality 4 subset of $U \cdot V$. Hence, since $|U \cdot V| = 4$, it follows that

$$U \cdot V = \{\pm 1, \pm y\}.$$

Thus it follows in view of (4) that $y - 1 \in \{\pm 1, \pm y\}$, whence $y \notin \{0, 1\}$ implies either $y = 2$ or $y = \frac{p+1}{2}$. However, in either case V is an arithmetic progression, contrary to assumption. So it remains to handle the case when neither U nor V is an arithmetic progression, and hence $x, y \notin \{-1, 0, 1, 2, \frac{p+1}{2}\}$.

If (5) and (7) hold, then $x = 2$, while if (6) and (7) hold, then $y \notin \{0, 1\}$ implies $x = -1$. Both cases contradict that $x \notin \{-1, 0, 1, 2, \frac{p+1}{2}\}$.

Suppose (5) and (6) hold. Then $x \neq y$ implies that x and y are the two distinct roots of $z^2 - z + 1$. Moreover, (4) implies that $U \cdot V = \{1, 2, x, y\}$, whence (6) gives

$$\{1, 2, 3, 4, x + 1, x + 2, y + 1, y + 2\} \subseteq U \cdot V + U \cdot V. \quad (8)$$

Suppose $|U \cdot V + U \cdot V| \leq 7$. Then at least one of the following cases holds: $x \in \{-1, 0, 1, 2, 3\}$, $y \in \{-1, 0, 1, 2, 3\}$, $x = y$, $x = y + 1$ or $y = x + 1$. If $x = y + 1$ or $y = x + 1$, say $x = y + 1$, then (6) implies $y = 0$, a contradiction. Consequently, since $x, y \notin \{-1, 0, 1, 2, \frac{p+1}{2}\}$ and $x \neq y$, we conclude that either $x = 3$ or $y = 3$, say $y = 3$. Hence (6) implies $x = -2$, whence (5) yields $7 = 0$, contradicting that $p > 7$. So we conclude that $|U \cdot V + U \cdot V| \geq 8 > |U \cdot V| + |U \cdot V| - 1$, and thus that $U \cdot V$ cannot be an arithmetic progression, as desired.

From the previous two paragraphs, we conclude that at most one of (5), (6) and (7) can hold, and thus that at least two of the quantities xy , $x + y$ and $xy + 1$ are not contained in $\{1, x, y\}$. Since $|U \cdot V| = 4$, all the quantities xy , $x + y$ and $xy + 1$ not contained in $\{1, x, y\}$ must be equal. Thus, as $xy + 1 \neq xy$, we see that $\{1, x, y\} \cap \{xy, x + y, xy + 1\}$ is nonempty. Hence, if $xy = x + y$ are the two quantities outside $\{1, x, y\}$, then (7) holds and so $x + y = xy = x - 1$, contradicting that $y \notin \{-1, 0, 1, 2, \frac{p+1}{2}\}$; while if $x + y = xy + 1$ are the two quantities outside $\{1, x, y\}$, then (5) holds and so $x + y = xy + 1 = 2$, which when combined with (5) yields $(y - 1)^2 = 0$, contradicting that $y \notin \{0, 1\}$. This completes the proof.

We now proceed with the proof of Theorem 1.

PROOF. First suppose $p > 7$ (we will handle the cases $p \leq 7$ afterwards), and assume by contradiction that $|S_1 \cdot S_2| \leq p - 1$. Let $h := \lfloor \frac{2p+1}{5} \rfloor$ and let $S_1 = U_1 \cdots U_h$ and $S_2 = V_1 \cdots V_h$ be factorizations of S_1 and S_2 into square-free subsequences U_i and V_i such that $|U_i| = |V_i| = 3$ for $i \leq p - 2h$ and $|U_i| = |V_i| = 2$ for $i > p - 2h$ (such factorizations are easily seen to exist in view of $\max\{h(A), h(B)\} \leq h$, $\frac{p}{3} < h < \frac{p}{2}$ and $|S_1| = |S_2| = p$; see for instance [1]). Let $A_i := U_i \cdot V_i$ for $i = 1, \dots, h$. Note $|A_i| = 2$ for $i \geq p - 2h + 1$ and that Lemma 1 implies $|A_i| \geq 4$ for $i \leq p - 2h$, with $|A_i| = 4$ possible only if A_i is not an arithmetic progression. Also, $\sum_{i=1}^h A_i \subseteq S_1 \cdot S_2$, and hence $|S_1 \cdot S_2| \leq p - 1$ implies

$$\left| \sum_{i=1}^h A_i \right| \leq p - 1. \quad (9)$$

Thus, in view of Theorem 2 applied to $\sum_{i=2}^h A_i + A_1$, we conclude that

$$\left| \sum_{i=2}^h A_i \right| \leq p - 4. \quad (10)$$

Since $h > \frac{p}{3}$ (in view of $p \geq 11$), it follows that $|A_h| = 2$, and thus A_h is an arithmetic progression. Iteratively applying Theorem 2 to

$$A_h + A_{h-1}, (A_h + A_{h-1}) + A_{h-2}, \dots, \sum_{i=p-2h+2}^h A_i + A_{p-2h+1},$$

we conclude, in view of (9) and Corollary 1, that

$$\left| \sum_{i=p-2h+1}^h A_i \right| \geq 3h - p + 1, \quad (11)$$

with equality possible only if $\sum_{i=p-2h+1}^h A_i$ is an arithmetic progression.

Since $|A_i| \geq 4$ for $i \leq p-2h$, with $|A_i| = 4$ possible only if A_i is not an arithmetic progression, then iteratively applying Theorems 2 and 3 to

$$\sum_{i=p-2h+1}^h A_i + A_{p-2h}, \sum_{i=p-2h}^h A_i + A_{p-2h-1}, \dots, \sum_{i=2}^h A_i + A_1,$$

yields, in view of (9), (10) and (11) (note in the last application we may be forced to apply Theorem 2 instead of Theorem 3 even if $|A_h| = 4$), that

$$\left| \sum_{i=1}^h A_i \right| \geq 3h - p + 1 + 4(p - 2h) - 1 = 3p - 5h.$$

Thus (9) implies that $h \geq \frac{2p+1}{5}$, whence $h = \frac{2p+1}{5}$. Hence (11) gives $|\sum_{i=p-2h+1}^h A_i| \geq 3$.

Consequently, since $|A_i| \geq 4$ for $i \leq p-2h$, with $|A_i| = 4$ possible only if A_i is not an arithmetic progression, then iteratively applying Theorems 2 and 3 to

$$\sum_{i=p-2h+1}^h A_i + A_{p-2h}, \sum_{i=p-2h}^h A_i + A_{p-2h-1}, \dots, \sum_{i=3}^h A_i + A_2,$$

yields, in view of (10), (11) and Proposition 1, that

$$\left| \sum_{i=2}^h A_i \right| \geq 3h - p + 1 + 4(p - 2h - 1) = 3p - 5h - 3, \tag{12}$$

with equality possible only if $\sum_{i=2}^h A_i$ is an arithmetic progression. Note $h < \frac{p}{2}$ implies that $|A_1| \geq 4$ with equality possible only if A_1 is not an arithmetic progression. Thus, if equality holds in (12), then (9), Theorem 2 and Corollary 1 imply that

$$\left| \sum_{i=1}^h A_i \right| \geq 3p - 5h - 3 + 4 = 3p - 5h + 1; \tag{13}$$

while on the other hand, if the inequality in (12) is strict, then (13) follows from Theorem 2 and (9). Therefore we may assume (13) holds regardless, whence (9) implies that $h \geq \frac{2p+2}{5}$, a contradiction. This completes the proof for $p > 7$.

Suppose $p = 7$ and that $\max\{h(S_1), h(S_2)\} \leq 2$. Let $S_1 = U_1U_2$ and $S_2 = V_1V_2$ be factorizations of S_1 and S_2 into square-free subsequences U_i and V_i such that $|U_2| = |V_2| = 3$ and $|U_1| = |V_1| = 4$ (as before, such factorizations are easily seen to exist in view of $\max\{h(S_1), h(S_2)\} \leq 2$ and $|S_1| = |S_2| = p = 7$). Let $A_i = U_i \cdot V_i$, and note in view of Lemma 1 that $|A_i| \geq 4$ for $i = 1, 2$. Thus applying Theorem 2 to $A_1 + A_2$ implies $|A_1 + A_2| = 7 = p$, so that the proof is complete in view of $A_1 + A_2 \subseteq S_1 \cdot S_2$. The case $p = 5$ follows by a near identical argument, concluding the proof.

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