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Some Discrete Representations Of q-Classical Linear Forms*

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Abstract

We give a discrete measure for some H_q-classical forms and some consequent summation formulas.

1 Introduction and Preliminaries

In [4], H_q-classical orthogonal polynomials are exhaustively described and integral or discrete representations of corresponding regular forms are given, except in some cases where the problem remains open (see also [3] for the H_q-semiclassical case). So, the aim of this contribution is to establish discrete representations of two canonical situations in [4] which are the q-analogous of Hermite (for $0 < q < 1, q > 1$) and the q-analogous of Laguerre (for $q > 1$).

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, for any $f \in \mathcal{P}$, we let fu , be the form defined by duality $\langle fu, p\rangle := \langle u, fp\rangle$, $p \in \mathcal{P}$.
Let $\langle \delta_c, p\rangle = p(c)$, $c \in \mathbb{C}$, $p \in \mathcal{P}$.

The form u is called regular if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ of monic polynomials, $\deg P_n = n$, $n \geq 0$ such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m} \ , \ n, m \geq 0 \ ; \ r_n \neq 0 \ , \ n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u and fulfils the standard recurrence relation:

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0 \end{cases} \tag{1}$$

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with $\gamma_{n+1} \neq 0$, $n \geq 0$.

The form u is said to be normalized if $(u)_0 = 1$ where in general $(u)_n = \langle u, x^n \rangle, n \geq 0$, are the moments of u. In this paper we suppose that any form will be normalized.

Let us introduce the Hahn's operator

$$(H_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x} \ , \ f \in \mathcal{P} \ , \ q \in \widetilde{\mathbb{C}},$$

where $\widetilde{\mathbb{C}} := \mathbb{C} - \Big(\{0\} \cup \big(\underset{n \geq 0}{\cup} \{z \in \mathbb{C},\, z^n = 1\} \big) \Big)$.
By duality we have

$$\langle H_q u, f \rangle = -\langle u, H_q f \rangle,\ u \in \mathcal{P}',\ f \in \mathcal{P}.$$

DEFINITION. A form u is called H_q- classical when it is regular and there exists two polynomials ϕ (monic) and ψ with $\deg(\phi) \leq 2$, $\deg(\psi) = 1$ such that

$$H_q(\phi u) + \psi u = 0. \tag{2}$$

The corresponding orthogonal sequence $\{P_n\}_{n \geq 0}$ is called H_q-classical.

We are going to use the following notations and results [1,2,5]

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ \prod\limits_{k=0}^{n-1} (1 - aq^k),\ n \geq 1. \end{cases} \tag{3}$$

$$(a;q)_n = (-1)^n a^n q^{\frac{n(n-1)}{2}} (a^{-1}; q^{-1})_n,\ n \geq 0,\ a, q \neq 0. \tag{4}$$

$$(a;q)_\infty = \prod_{k=0}^{+\infty} (1 - aq^k),\ \mid q \mid < 1. \tag{5}$$

$$(a;q)_n = \begin{cases} \dfrac{(a;q)_\infty}{(aq^n;q)_\infty},\ \mid q \mid < 1, \\[2ex] \dfrac{(aq^{-1}q^n;q^{-1})_\infty}{(aq^{-1};q^{-1})_\infty},\ \mid q \mid > 1. \end{cases} \tag{6}$$

$$(z;q)_\infty = \sum_{k=0}^{+\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k} z^k,\ \mid q \mid < 1. \tag{7}$$

$$\frac{1}{(z,q)_\infty} = \sum_{k=0}^{+\infty} \frac{1}{(q;q)_k} z^k,\ \mid q \mid < 1\,,\ \mid z \mid < 1. \tag{8}$$

2 Discrete measure for some H_q-classical forms

2.1

Consider the symmetric H_q-classical linear form u which is the q-analog of Hermite functional. We have [4]

$$\begin{cases} \beta_n = 0,\ n \geq 0, \\ \gamma_{n+1} = \dfrac{1-q^{n+1}}{2(1-q)} q^n,\ n \geq 0, \\ H_q(u) + 2xu = 0. \end{cases} \tag{9}$$

$$\langle u, f\rangle = \begin{cases} \dfrac{\sqrt{2}}{\pi}(q-1)^{1/2}\dfrac{(q^{-2};q^{-2})_\infty}{(q^{-1};q^{-2})_\infty}\displaystyle\int_{-\infty}^{+\infty} \dfrac{f(x)}{\left(-2(q-1)x^2;q^{-2}\right)_\infty}dx,\ f \in \mathcal{P},\ q>1, \\ K_1 \displaystyle\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}} \left(2q^2(1-q)x^2;q^2\right)_\infty f(x)dx,\ f\in\mathcal{P},\ 0<q<1, \end{cases} \tag{10}$$

with

$$K_1 = \frac{1}{2}\Big(\int_0^{+\frac{1}{q\sqrt{2(1-q)}}} \left(2q^2(1-q)x^2;q^2\right)_\infty dx\Big)^{-1}. \tag{11}$$

$$(u)_{2n} = \frac{1}{2^n}\frac{(q;q^2)_n}{(1-q)^n},\ \ (u)_{2n+1} = 0,\ n \geq 0. \tag{12}$$

PROPOSITION 1. We have the following discrete representations:
For $f \in \mathcal{P}$, $q > 1$

$$\langle u, f\rangle = \frac{1}{2(q^{-1};q^{-2})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k}\Big\{f\Big(\frac{-iq^k}{\sqrt{2(q-1)}}\Big) + f\Big(\frac{iq^k}{\sqrt{2(q-1)}}\Big)\Big\}. \tag{13}$$

For $f \in \mathcal{P}$, $0 < q < 1$

$$\langle u, f\rangle = \ 2^{-1}(q;q^2)_\infty \sum_{k=0}^{+\infty}\frac{q^k}{(q^2;q^2)_k}\Big\{f\Big(\frac{-q^k}{\sqrt{2(1-q)}}\Big) + f\Big(\frac{q^k}{\sqrt{2(1-q)}}\Big)\Big\}. \tag{14}$$

PROOF. Let $q > 1$ by (6), equation (12) becomes

$$(u)_{2n} = \frac{1}{2^n(1-q)^n}\frac{(q^{2n-1};q^{-2})_\infty}{(q^{-1};q^{-2})_\infty},\ n \geq 0.$$

On account of (7), we get

$$(u)_{2n} = \frac{1}{(q^{-1};q^{-2})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k}\Big(\frac{iq^k}{\sqrt{2(q-1)}}\Big)^{2n},\ n \geq 0.$$

Therefore

$$(u)_{2n} = \Big\langle \frac{1}{(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k} \delta_{\frac{iq^k}{\sqrt{2(q-1)}}}, x^{2n} \Big\rangle,\ n \geq 0.$$

But $(u)_{2n+1} = 0$, $n \geq 0$, yields to

$$(u)_n = \langle u, x^n \rangle = \Big\langle \frac{1}{2(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k} \Big\{ \delta_{\frac{-iq^k}{\sqrt{2(q-1)}}} + \delta_{\frac{iq^k}{\sqrt{2(q-1)}}} \Big\}, x^n \Big\rangle,\ n \geq 0.$$

Consequently

$$u = \frac{1}{2(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k} \Big\{ \delta_{\frac{-iq^k}{\sqrt{2(q-1)}}} + \delta_{\frac{iq^k}{\sqrt{2(q-1)}}} \Big\}.$$

Then we get the desired result (13).

When $0 < q < 1$, by virtue of (6), equation (12) becomes

$$(u)_{2n} = \frac{(q;q^2)_\infty}{2^n(1-q)^n(q^{2n+1};q^2)_\infty},\ n \geq 0,$$

on account of (8), it follows that

$$(u)_{2n} = (q;q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2;q^2)_k} \Big(\frac{q^k}{\sqrt{2(1-q)}} \Big)^{2n},\ n \geq 0.$$

Then

$$(u)_n = \Big\langle 2^{-1}(q;q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2;q^2)_k} \Big\{ \delta_{\frac{-q^k}{\sqrt{2(1-q)}}} + \delta_{\frac{q^k}{\sqrt{2(1-q)}}} \Big\}, x^n \Big\rangle,\ n \geq 0.$$

Consequently, we are lead to

$$u = 2^{-1}(q;q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2;q^2)_k} \Big\{ \delta_{\frac{-q^k}{\sqrt{2(1-q)}}} + \delta_{\frac{q^k}{\sqrt{2(1-q)}}} \Big\}. \tag{15}$$

Hence (14).

2.2

Consider the q-analogous of Laguerre linear form u given in [4,pp 68] .We have

$$\begin{cases} \beta_n = \{1-(1+q)q^n\}q^{n-1},\ n \geq 0, \\ \gamma_{n+1} = (q^{n+1}-1)q^{3n},\ n \geq 0, \\ H_q(xu) - (q-1)^{-1}(x+1)u = 0. \end{cases} \tag{16}$$

For $q > 1$, we have the following representations [4]:

$$\langle u, f\rangle = \begin{cases} (2\pi \ln q)^{-1/2} q^{-1/8} \displaystyle\int_{-\infty}^{0} \mid x \mid^{-3/2} \exp\Big(-\frac{\ln^2 \mid x \mid}{2\ln q}\Big) f(x)dx,\ f \in \mathcal{P}, \\ \displaystyle\sum_{k=0}^{+\infty} (-1)^k \frac{q^{-k^2} s(k)}{(q^{-1};q^{-1})_k} f(-q^k),\ f \in \mathcal{P}, \end{cases} \tag{17}$$

where

$$s(k) = \sum_{m=0}^{+\infty} \frac{q^{-(\frac{1}{2}m(m+1)+km)}}{(q^{-1};q^{-1})_m} (u)^{\phi}_{m+k},\ k \geq 0, \tag{18}$$

and $(u)^{\phi}_{2n} = (q-1)^n$, $(u)^{\phi}_{2n+1} = 0$, $n \geq 0$.

The moments of u are given by the following formulas:

$$(u)_n = (-1)^n q^{\frac{1}{2}n(n-1)},\ n \geq 0. \tag{19}$$

PROPOSITION 2. The form u possesses the following discrete representation: For $f \in \mathcal{P}$, $q > 1$

$$(-1;q^{-1})_\infty(-q^{-1};q^{-1})_\infty \langle u, f\rangle = \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} \sum_{\mu=0}^{k} \frac{q^{-\mu^2+(k-1)\mu}}{(q^{-1};q^{-1})_\mu (q^{-1};q^{-1})_{k-\mu}} f\big(-q^{2\mu-k}\big), \tag{20}$$

PROOF. From (4), for (19) we obtain

$$(u)_n = (-1)^n \frac{(-1;q)_n}{(-1;q^{-1})_n},\ n \geq 0. \tag{21}$$

Let $q > 1$, taking (6) into account, equation (21) can be written in the following way

$$(u)_n = \frac{(-1)^n}{(-1;q^{-1})_\infty(-q^{-1};q^{-1})_\infty} (-q^{n-1};q^{-1})_\infty (-q^{-n};q^{-1})_\infty,\ n \geq 0.$$

In accordance of (7), we get

$$(u)_n = \frac{(-1)^n}{(-1;q^{-1})_\infty(-q^{-1};q^{-1})_\infty} \sum_{k=0}^{+\infty} \frac{q^{-\frac{k(k-1)}{2}}}{(q^{-1};q^{-1})_k} q^{k(n-1)} \sum_{k=0}^{+\infty} \frac{q^{-\frac{k(k-1)}{2}}}{(q^{-1};q^{-1})_k} q^{-kn},\ n \geq 0.$$

Using the Cauchy product, the last expression becomes (for $n \geq 0$)

$$(u)_n = \frac{1}{(-1;q^{-1})_\infty(-q^{-1};q^{-1})_\infty} \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} \sum_{\mu=0}^{k} \frac{q^{-\mu^2+(k-1)\mu}}{(q^{-1};q^{-1})_\mu (q^{-1};q^{-1})_{k-\mu}} \big(-q^{2\mu-k}\big)^n.$$

Then, the discrete measure in (20) is deduced.

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