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Stability Of Periodic Solutions In Extended Gierer-Meinhardt Model*

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Abstract

We employ coincidence degree method to prove existence of T-periodic solutions in $\mathcal{D}$ for extended Gierer-Meinhardt (EG-M) model, where $\mathcal{D}$ is a strictly positively invariant region. Furthermore, Floquet theory is provided to analyze uniqueness of a T-periodic solution $x_0(t)$ in $\mathcal{D}$ and stability of $x_0(t)$ is presented.

1 Introduction

Gierer-Meinhardt model [6, p. 376-380] is of form:

$$\begin{cases} \dot{u} = a(1 - bu + c\frac{u^2}{v}) \\ \dot{v} = d(u^2 - ev). \end{cases}$$

If we set the constants a, b, c, d and e to be positive continuous T-periodic functions of t with period $T > 0$, then the corresponding model is called the extended Gierer-Meinhardt (EG-M) model.

We focus on the dynamics of periodic solutions of EG-M model. Let

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{and} \quad F(t, x(t)) = \begin{pmatrix} a(t)(1 - b(t)u(t) + c(t)\frac{u^2(t)}{v(t)}) \\ d(t)(u^2(t) - e(t)v(t)) \end{pmatrix}.$$

Then EG-M model is defined by

$$\dot{x}(t) = F(t, x(t)) \tag{1}$$

with conditions

$$a(t) > 0, \;\; 1.8 < b(t) < 2, \;\; 0.01 < c(t) < 0.1, \;\; d(t) > 0, \;\; \frac{1}{2} < e(t) < 1. \tag{2}$$

LEMMA 1. There exists a strictly positively invariant region

$$\mathcal{D} = \left\{(u, v) \in \mathbb{R}^2 \, : \, \frac{1}{2} \le u \le 0.7, \;\; \frac{1}{4} \le v \le 1\right\}$$

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for EG-M model given by (1) with conditions (2).

PROOF. Clearly $\mathcal{D}$ is a closed convex subset of $\mathbb{R}^2$. We only need to check whether $n(u,v)\cdot F(t,(u,v))<0$ along the boundaries of $\mathcal{D}$, where $n(u,v)$ is the unit normal vector field along the boundary of $\mathcal{D}$ and $F(t,(u,v))$ is defined in (1). Notice that for any $(u,v)\in\mathcal{D}$,

$$\frac{1}{2}\le u\le 0.7,\ \ \frac{1}{4}\le v\le 1. \tag{3}$$

Let $l_1=\{(u,v)\in\mathbb{R}^2\,:\,u=\frac{1}{2}\}$, for any $(u,v)\in l_1\cap\partial\mathcal{D}$, $n(u,v)=(-1,0)$ and

$$F(t,(u,v))=(a(t)(1-b(t)u(t)+c(t)\frac{u^2(t)}{v(t)}),d(t)(u^2(t)-e(t)v(t))),$$

by (2) and (3),

$$n(u,v)\cdot F(t,(u,v))=-a(t)(1-\frac{1}{2}b(t)+c(t)\frac{\frac{1}{4}}{v(t)})<0.$$

Let $l_2=\{(u,v)\in\mathbb{R}^2\,:\,u=0.7\}$, for any $(u,v)\in l_2\cap\partial\mathcal{D}$, $n(u,v)=(1,0)$. It follows from (2) and (3) that

$$n(u,v)\cdot F(t,(u,v))=a(t)(1-0.7b(t)+c(t)\frac{0.49}{v(t)})<0.$$

Let $l_3=\{(u,v)\in\mathbb{R}^2\,:\,v=\frac{1}{4}\}$, for any $(u,v)\in l_3\cap\partial\mathcal{D}$, $n(u,v)=(0,-1)$. From (2) and (3), we get

$$n(u,v)\cdot F(t,(u,v))=-d(t)(u^2(t)-\frac{1}{4}e(t))<0.$$

Let $l_4=\{(u,v)\in\mathbb{R}^2\,:\,v=1\}$, for any $(u,v)\in l_4\cap\partial\mathcal{D}$, $n(u,v)=(0,1)$. It follows from (2) and (3),

$$n(u,v)\cdot F(t,(u,v))=d(t)(u^2(t)-e(t))<0.$$

Since $n(u,v)\cdot F(t,(u,v))<0$ for all $(u,v)\in\partial\mathcal{D}$, $\mathcal{D}$ is a strictly positively invariant region.

Linearize the system (1) with respect to its a T-periodic solution $x(t)=(u(t),v(t))^T\in\mathcal{D}$ for any $t\in\mathbb{R}$ (if such a T-periodic solution exists), then we get

$$\dot{W}(t)=A(t)W(t), \tag{4}$$

where

$$A(t)=F'_{x(t)}=\begin{pmatrix} -a(t)b(t)+\frac{2a(t)c(t)u(t)}{v(t)} & -\frac{a(t)c(t)u^2(t)}{v^2(t)} \\ 2d(t)u(t) & -d(t)e(t)\end{pmatrix}.$$

PROPOSITION 2. Linear system (4) satisfies $tr(A(t))<0$ and $det(A(t))>0$ for any $t\in\mathbb{R}$.

PROOF. By (2) and (3),

$$tr(A(t)) = -a(t)b(t) + \frac{2a(t)c(t)u(t)}{v(t)} - d(t)e(t) < 0.$$

Furthermore,

$$\begin{aligned}\det(A(t)) &= a(t)b(t)d(t)e(t) - \frac{2a(t)c(t)d(t)e(t)u(t)}{v(t)} + \frac{2a(t)c(t)d(t)u^3(t)}{v^2(t)} \\ &> a(t)d(t)\left(b(t)e(t) - \frac{2c(t)e(t)u(t)}{v(t)}\right) > 0.\end{aligned}$$

Now, let us state the main result:

THEOREM 3. For EG-M model with conditions (2), there exists only one T-periodic solution $x_0(t)$ in $\mathcal{D}$, and $x_0(t)$ is locally uniformly asymptotically stable.

2 Preliminary

Consider the nonlinear system

$$\dot{x}(t) = V(t, x(t)), \;\; x(t_0) = x_0, \;\; x(t) \in \mathbb{R}^2. \tag{5}$$

LEMMA 4. If $x^*(t)$ is an exponentially stable solution of (5), then it is also a uniformly asymptotically stable solution of (5).

For proof, see [3, p. 178-179].

Let $\mathcal{X} = \{x \in C([0,T]) \,|\, x(0) = x(T)\}$. Clearly $\mathcal{X}$ is a Banach space with the supremum norm. Define $Lx(t) = \dot{x}(t)$ with domain

$$Dom(L) = \{x \in C^1([0,T]) \,|\, x(0) = x(T)\}.$$

It is easy to verify that $Dom(L)$ is contained in $\mathcal{X}$, the range of L is $Im(L) = \{z(t) \in \mathcal{X} \,|\, \int_0^T z(t)dt = 0\}$ and L is a Fredholm mapping of index 0. Let

$$\Theta = \{x \in Dom(L) \,|\, x(t) \in \mathcal{D}, \quad \forall t \in [0,T]\}. \tag{6}$$

Define $\mathcal{F}_1 : \Theta \to \mathcal{X}$ by $\mathcal{F}_1(x) = F(\cdot, x(\cdot))$ and $H_1(x)(t) = \mathcal{F}_1(x)(t) - Lx(t)$.

Now, construct a homotopy family

$$H_\lambda : (Dom(L) \cap \Theta) \times [0,1] \to \mathcal{X}$$

to be of the form

$$H_\lambda(x)(t) = \mathcal{F}_\lambda(x)(t) - Lx(t), \tag{7}$$

where $\mathcal{F}_\lambda : \Theta \times [0,1] \to \mathcal{X}$ with

$$\mathcal{F}_\lambda(x)(t) = \begin{pmatrix} a(t)(1 - \widetilde{b}(t)u(t) + \widetilde{c}(t)\frac{u^2(t)}{v(t)}) \\ d(t)(u^2(t) - \widetilde{e}(t)v(t)) \end{pmatrix}. \tag{8}$$

Here $\tilde{b}(t) = 1.9(1-\lambda)+\lambda b(t)$, $\tilde{c}(t) = 0.05(1-\lambda)+\lambda c(t)$ and $\tilde{e}(t) = 0.8(1-\lambda)+\lambda e(t)$ with $\lambda \in [0,1]$. It is easy to verify that $\mathcal{F}_\lambda : \overline{\Theta} \times [0,1] \to \mathcal{X}$ is L-compact. For more details of degree theory, see [5, Ch. I–IV].

LEMMA 5. Given $\lambda \in [0,1]$, if $x_\lambda(t) \in \Theta$ is a T-periodic solution of the system

$$\dot{x}(t) = \mathcal{F}_\lambda(x)(t), \tag{9}$$

then $\partial\mathcal{D}$ is an a priori bound of $x_\lambda(t)$.

PROOF. Clearly, $\tilde{b}(t), \tilde{c}(t)$ and $\tilde{e}(t)$ satisfy conditions (2). System (9) is an EG-M model. By Lemma 1, $\mathcal{D}$ is still a strictly positively invariant region of system (9). None of T-periodic solutions of (9) in Θ can touch the boundary of $\mathcal{D}$.

COROLLARY 6. $0 \not\in H_\lambda((Dom(L) \cap \partial\mathcal{D}) \times [0,1])$.

LEMMA 7. $D_L(H_0(x)(t), \Theta) = D_B(H_0(x)(t), \mathcal{D}) = 1$, where D_L denote Leray-Schauder degree and D_B denote Brouwer degree.

PROOF. For the system $H_0(x)(t) = 0$, there is only one steady-state

$$p = \left(\frac{1.04}{1.9}, \left(\frac{1.04}{1.9}\right)^2 \frac{1}{0.8}\right)$$

in the strictly positively invariant region $\mathcal{D}$, which is a trivial T-periodic solution. Since $H_0(x)(t) = 0$ is an autonomous system, Proposition 2 and Bendixson's Criteria guarantee that p is only one T-periodic solution in $\mathcal{D}$.

For the system $H_0(x)(t) = 0$, Leray-Schauder degree of H_0 in $\mathcal{D}$ is in fact reduced into Brouwer degree. Therefore, by Proposition 2,

$$D_L(H_0(x)(t), \Theta) = D_B(H_0(x)(t), \mathcal{D}) = D_B(\mathcal{F}_0(x)(t), \mathcal{D}) = sign(\det A_1(t)) = 1,$$

where

$$A_1(t) = \begin{pmatrix} -1.9a + \frac{0.1au(t)}{v(t)} & -\frac{0.05au^2(t)}{v^2(t)} \\ 2du(t) & -0.8d \end{pmatrix}.$$

For more details, see the similar proof in [4].

LEMMA 8. For system (4) with conditions (2), zero is the only T-periodic solution.

PROOF. Suppose (4) has a non-trivial T-periodic solution called $W_1(t)$. By Proposition 2 and Floquet theory [2, p. 93-105], its orbit Γ is an orbitally asymptotically stable. For $s \in \mathbb{R}$, $sW_1(t)$ is also a T-periodic solution of (4). Then orbit of $sW_1(t)$ can not be attracted to Γ for any $s \in \mathbb{R}$. This leads a contradiction to the orbital asymptotic stability of Γ.

REMARK 9. For the linear system (4), if $tr(A(t))$ does not change sign in some simply connected region $E \subset \mathbb{R}^2$, then (4) has no non-trivial periodic solution in E; since the system (4) is a linearization of an non-autonomous system, Bendixson's Criteria can not be used to prove Lemma 8.

LEMMA 10. Suppose $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a completely continuous map of a Banach space such that $\mathcal{F}(0) = 0$ and $\mathcal{F}$ is Frechet differentiable at 0 with Frechet derivative $T \in K(\mathcal{X})$, where $K(\mathcal{X})$ is a set of all compact operators defined on $\mathcal{X}$. If $I - T \in L(\mathcal{X})$

is regular (invertible), then there exists $\eta > 0$ such that, for $B = \{x \in \mathcal{X} : \|x\|_\infty < \eta\}$, we have

$$D(\mathcal{F} - I, B) = D(T - I, B).$$

For proof, see [1, Ch. 14].

Assume system (4) is the linearization of system (1) with respect to $x_0(t)$, by Theorem 2.10 of [2, p.97], system (4) can be transformed into an autonomous system

$$\dot{Z}(t) = RZ(t), \tag{10}$$

where R is called a Monodromy matrix of $A(t)$.

LEMMA 11. Let $A(t)$ and $W(t)$ be defined in (4), $LW(t) = \dot{W}(t)$; $B(x_0(t), \epsilon) \subset \Theta$ denote a small neighborhood of $x_0(t)$, $B(0, \epsilon) \subset \Theta \backslash \{x_0(t)\}$ denote a small neighborhood of 0. Set

$$Q(W(t), \lambda) = (\lambda R + (1 - \lambda)A(t))W(t) - LW(t),$$

then

$$D_B(Q(\cdot, 1), B(0, \epsilon) \cap \mathbb{R}^2) = 1.$$

PROOF. Clearly, $tr(R) = \rho_1 + \rho_2 = \dfrac{1}{T}\int_0^T tr(A(s))ds(mod\dfrac{2\pi i}{T}) < 0$, where ρ_1 and ρ_2 are eigenvalues of R. By Proposition 2, for any $\lambda \in [0, 1]$,

$$tr(\lambda R + (1 - \lambda)A(t)) = \lambda tr(R) + (1 - \lambda)tr(A(t)) < 0,$$

and Remark 9 implies that $Q(W(t), \lambda) = 0$ has only one trivial T-periodic solution in $B(0, \epsilon)$. By degree invariance with respect to homotopy family,

$$D_L(Q(\cdot, 0), B(0, \epsilon)) = D_L(Q(\cdot, 1), B(0, \epsilon)) = D_B(Q(\cdot, 1), B(0, \epsilon) \cap \mathbb{R}^2).$$

Consider the Taylor expansion of $H_1(x)(t)$ at $x_0(t) \in B(x_0(t), \epsilon)$, where $H_1(x)(t)$ is defined in (7) as $\lambda = 1$. Then we have

$$H_1(x)(t) = H_1(x_0)(t) + M(t)(x(t) - x_0(t)) + h(t, x(t) - x_0(t)),$$

where $M = \mathcal{F}_1' - L$ is $(H_1)_x'$ and $h(t, x(t) - x_0(t))$ is a function of $o(\|x(t) - x_0(t)\|_\infty)$. Since $x_0(t)$ is the unique solution of $H_1(x)(t) = 0$ in $\mathcal{D}$, by excision property of the degree, $D_L(H_1(x)(t), \Theta) = D_L(H_1(x)(t), B(x_0(t), \epsilon))$ and by Lemma 10,

$$D_L(H_1(x)(t), B(x_0(t), \epsilon)) = D_L(M(t)(x(t) - x_0(t)), B(x_0(t), \epsilon)).$$

Let $W(t) = x(t) - x_0(t)$, then by Lemma 7,

$$\begin{aligned} D_L(H_0(x)(t), \Theta) &= D_L(H_1(x)(t), \Theta) \\ &= D_L(M(t)W(t), B(0, \epsilon)) = D_L(Q(\cdot, 0), B(0, \epsilon)) \\ &= D_L(Q(\cdot, 1), B(0, \epsilon)) = D_B(Q(\cdot, 1), B(0, \epsilon) \cap \mathbb{R}^2) = 1. \end{aligned}$$

3 Proof of Theorem 3

PROOF. (**Existence**) Combine Lemma 5, Corollary 6 and Lemma 7, by a general existence theorem of the Leray-Schauder type, we get

$$D_L(H_1(x)(t), \Theta) = D_L(H_0(x)(t), \Theta) = D_B(\mathcal{F}_0(x)(t), \mathcal{D}) = 1,$$

which implies that there at least exists one T-periodic solution $x_0(t) = (u_0(t), v_0(t))^T$ of EG-M model in $\mathcal{D}$. If a, b, c, d and e are constants, it is easy to show that there is only one trivial T-periodic solution $x_0 \in int(\mathcal{D})$, otherwise, we can easily verify that $x_0(t)$ is a nontrivial T-periodic solution of EG-M model in $\mathcal{D}$ by substituting $x_0(t)$ into EG-M model.

(**Uniqueness**) Define $C_T = \{x(t) \in \Theta | x(t)$ satisfies (1) with conditions (2)$\}$. Since $x_0(t) \in C_T$, C_T is not an empty set. If a, b, c, d and e are constants, there is only one constant solution in C_T.

If one of $a(t), b(t), c(t), d(t)$ and $e(t)$ is a non-trivial T-periodic function, then $x_0(t) \in C_T$ is a non-trivial T-periodic solution. Assume C_T is not a singleton; we pick

$$x_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \ x_2(t) = \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix},$$

in C_T and substitute them into (1) to get

$$\dot{x_i}(t) = F(t, x_i(t)), \quad i = 1, 2. \tag{11}$$

Define $z(t) = x_1(t) - x_2(t)$. By the mean value theorem, we get

$$\dot{z}(t) = z(t) \int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta, \tag{12}$$

and

$$\int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta = \begin{pmatrix} -a(t)b(t) + 2a(t)c(t)m(t) & -a(t)c(t)n(t) \\ 2u(t)d(t) & -d(t)e(t) \end{pmatrix},$$

where

$$m(t) = \int_0^1 \frac{u_2(t) + \theta(u_1(t) - u_2(t))}{v_2(t) + \theta(v_1(t) - v_2(t))} d\theta,$$

$$n(t) = \int_0^1 \frac{[u_2(t) + \theta(u_1(t) - u_2(t))]^2}{[v_2(t) + \theta(v_1(t) - v_2(t))]^2} d\theta.$$

By (3)

$$m(t) \le 2(u_1(t) + u_2(t)) \le 2.8.$$

From (2), it follows

$$tr\left(\int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta\right) = -d(t)e(t) - a(t)b(t) + 2a(t)c(t)m(t) < 0.$$

This implies that the zero solution is the only one T-periodic solution for (12) by Remark 9. Hence, $x_1(t) = x_2(t)$. C_T is a singleton.

(**Stability**) If $a(t), b(t), c(t), d(t)$ and $e(t)$ are constant functions, then $x_0 \in \mathcal{D}$ is a constant solution of system (1). By Proposition 2, x_0 is a locally uniformly asymptotically stable solution.

If one of $a(t), b(t), c(t), d(t)$ and $e(t)$ is a non-trivial T-periodic function, $x_0(t)$ is a non-trivial T-periodic solution of system (1). By Proposition 2, one Floquet exponent ρ_1 has negative real part. If $Real(\rho_2) < 0$, $x_0(t)$ is locally uniformly asymptotically stable by Theorem 2.13 of [2, p.101] and Lemma 4.

To show that $Real(\rho_2) < 0$ always holds by treating these two cases: (1) ρ_2 is a complex number. Notice that ρ_1 and ρ_2 are conjugate eigenvalues of R. Thus $Real(\rho_2) < 0$; (2) ρ_2 is a real number. Clearly ρ_1 is also a real number and $\rho_1 < 0$ implies that $\rho_2 \neq 0$ (otherwise, it is a contradicts Lemma 8. If $\rho_2 > 0$, then $\det(R) < 0$, $sign(\det(R)) = -1 = D_B(Q(\cdot, 1), B(0, \epsilon) \cap \mathbb{R}^2)$, which contradicts Lemma 11.

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