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The Number Of Spanning Trees And Chains Of Graphs*

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Abstract

Let $t(G)$ denote the number of spanning trees of a graph G. A *chain* of two connected vertices $u, v(d_G(u), d_G(v) \geq 3)$ in G, denoted by L_k, is defined as a path of G and $d_G(p) = 2$ for all $p \in V(L_k) - \{u, v\}$, where k is the length of the path. In this paper, we investigate the relationship between $t(G)$ and L_k of a graph G. In particular, the relationship between $t(G)$ and L_k of τ-optimal graph G is considered.

1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite connected graphs only. A *spanning subgraph* of a graph $G = (V, E)$ is a subgraph with vertex set V. A *spanning tree* is a spanning subgraph that is a tree. Let $\Gamma(n, m)$ denote the collection of all n vertices m edges graphs with no loops. Let $t(G)$ denote the number of spanning trees of a graph G. Spanning trees have been found to be structures of paramount importance in both theoretical and practical problems. As a result the number of spanning trees of a connected graph has been the focus for extensive attention in graph theoretical research.

A graph $G \in \Gamma(n, m)$ is called τ-optimal if $t(G) \geq t(H)$ for all $H \in \Gamma(n, m)$. An open extremal problem, with applications to the synthesis of reliable networks, is the characterization of τ-optimal graphs [1, 3, 4, 5, 6, 7]. In [5], authors introduced a lower bound for the trace of the k-th power of the Laplacian matrix of a graph in terms of its degree sequence. Using this inequality they developed an upper bound for the number of spanning trees of a graph in terms of the degree sequence of its completment that is sharp for, and only for, complete multipartite graphs. In [6], authors develop a powerful refinement of the upper bounding technique for the number of spanning trees. The improved bound yields a new technique to characterize many hitherto unknown types of τ-optimal graphs.

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We consider the reliability of graphs for which edges fail independently of each other with a constant probability q. A standard formula for the reliability of a graph G is

$$R(G,q)=\sum_{i=n-1}^{m} N_i(G)q^{m-i}(1-q)^i,$$

where $N_i(G)$ denotes the number of connected spanning subgraphs of G with i edges. Clearly $N_{n-1}(G)=t(G)$. Suppose that $G,H\in\Gamma(n,m)$. We have

$$\begin{aligned} R(G,q)-R(H,q) &= q^{n-m-1}(1-q)^{1-n} \\ &\quad\times\left[t(G)-t(H)+\sum_{i=n}^{m}(N_i(G)-N_i(H))q^{n-i-1}(1-q)^{i-n+1}\right]. \end{aligned}$$

If $t(G)>t(H)$, then $R(G,q)>R(H,q)$ for $q\to 1$. Thus τ-optimal graphs are uniformly most reliable in $\Gamma(n,m)$ for $q\to 1$.

In this paper, we investigate the relationship between the number of spanning trees and chains of a graph. In particular, the relationship between the number of spanning trees and chains of τ-optimal graphs is considered.

2 Number of Spanning Trees and Chains of Graphs

A *chain* of two connected vertices u,v $(d_G(u),d_G(v)\geq 3)$ in G, denoted by L_k, is defined as a path of G and $d_G(p)=2$ for all $p\in V(L_k)-\{u,v\}$, where k is the length of the path. If $k=1$, then L_1 is trivial, i.e., an edge. Two chains L_{k_1},L_{k_2} are said to be parallel if L_{k_1},L_{k_2} meet only in two common endpoints. Let $G-L_k=G[V(G)-V(L_k)+\{u,v\}]$ and $G/L_k=((G-L_k)+uv)/uv$, where u,v are two endpoints of L_k.

THEOREM 1. Let $L_k(k\geq 1)$ be a chain of a graph G. Then $t(G-L_k)\leq t(G)$ and $t(G/L_k)\leq t(G)$.

PROOF. We prove $t(G)=kt(G-L_k)+t(G/L_k)$ first. Let u and v be end vertices of L_k and $G^*=G-L_k+uv$. Then

$$t(G^*)=t(G^*-uv)+t(G^*/uv).$$

Since every spanning tree of G^* that does not contain uv yields k spanning trees of G, each of which does not contain L_k, and conversely, $kt(G-L_k)$ is the number of spanning trees of G that does not contain L_k.

Now to each spanning tree T of G^* that contains uv, there corresponds a spanning tree T/L_k of G/L_k. This correspondence is clearly a bijection. Therefore $t(G/L_k)$ is precisely the number of spanning trees of G that contain L_k. It follows that

$$t(G)=kt(G-L_k)+t(G/L_k).$$

Since $t(G-L_k)\geq 0$ and $t(G/L_k)\geq 0$, it is easy to have $t(G-L_k)\leq t(G)$ and $t(G/L_k)\leq t(G)$.

THEOREM 2. Let $L_k(k>3)$ be a chain of a graph G and u,v are two endpoints of L_k. Suppose that L_k does not contain and cut edges of G and $w\in V(G)-V(L_k)$

with $wu, wv \notin E(G)$. We construct two chains $L_{k_1}(k_1 = \lfloor k/2 \rfloor), L_{k_2}(k_2 = k - k_1)$, such that w, u and w, v are two endpoints of L_{k_1}, L_{k_2}, respectively. Then we have

$$t(G) < t(G - L_k + \{L_{k_1}, L_{k_2}\}).$$

PROOF. Let $G^* = G - L_k + \{L_{k_1}, L_{k_2}\}$. By the proof of Theorem 1, we have $t(G) = kt(G - L_k) + t(G/L_k)$. Similarly, we have

$$\begin{aligned} t(G^*) &= t(G^* - L_{k_1} - L_{k_2})k_1k_2 + k_1t((G^*/L_{k_2}) - L_{k_1}) \\ &\quad + k_2t((G^*/L_{k_1}) - L_{k_2}) + t(G^*/L_{k_1}/L_{k_2}). \end{aligned}$$

Since $k_1 = \lfloor k/2 \rfloor, k_2 = k - k_1$ and $k > 3$, we have $k_1k_2 \geq k$. Let $\widetilde{G} = G - L_k + uv$ and $\overline{G} = G^* - L_{k_1} - L_{k_2} + uw$. Let T be a spanning tree of $\widetilde{G}$ which contains uv, then $T - uv + uw$ is a spanning tree of $\overline{G}$ which does not contain vw, which implies

$$t(G/L_k) = t((G^*/L_{k_1}) - L_{k_2}).$$

Combined with $t(G - L_k) = t(G^* - L_{k_1} - L_{k_2})$, $t(G^*/L_{k_1}/L_{k_2}) > 0$ and $t((G^*/L_{k_2}) - L_{k_1}) > 0$, we have $t(G) < t(G^*)$.

Let $\widetilde{G} = G - L_k + uv$, $\overline{G}_1 = G^* - L_{k_1} - L_{k_2} + uw$ and $\overline{G}_2 = G^* - L_{k_1} - L_{k_2} + vw$. Let T be a spanning tree of $\overline{G}$ which contains uv, then one of the following results holds:

(1) $T - uv + uw$ is a spanning tree of $\overline{G}_1$ which does not contain vw, which implies $t(G/L_k) = t((G^*/L_{k_1}) - L_{k_2})$;

(2) $T - uv + vw$ is a spanning tree of $\overline{G}_2$ which does not contain uw, which implies $t(G/L_k) = t((G^*/L_{k_2}) - L_{k_1})$.

Combined with $t(G - L_k) = t(G^* - L_{k_1} - L_{k_2})$, $t((G^*/L_{k_1}) - L_{k_2}) > 0$, $t((G^*/L_{k_2}) - L_{k_1}) > 0$ and $t(G^*/L_{k_1}/L_{k_2}) > 0$, we have $t(G) < t(G^*)$.

3 Number Of Spanning Trees and Chains of τ-Optimal Graphs

We have the following result.

THEOREM 3. Let G be a $\tau-$ optimal graph and $L_{k_1}(k_1 > 0), L_{k_2}(k_2 > 0)$ are two chains of G. Then

(a) $t((G - L_{k_1})/L_{k_2}) \leq t(G - L_{k_2})$, and

(b) $t((G - L_{k_1})/L_{k_2}) \leq t(G/L_{k_1})$.

PROOF of (a). We prove by contradiction. Let G be a τ-optimal graph, and assume that there are two chains L_{k_1} and L_{k_2} of G with

$$t((G - L_{k_1})/L_{k_2}) > t(G - L_{k_2}).$$

Let u and v be end vertices of L_{k_1}. We construct a new graph G^* from $(G - L_{k_1})/L_{k_2}$ by adding a chain L_k in $(G - L_{k_1})/L_{k_2}$, with u, v as end vertices and $k = k_1 + k_2$. Then we have $|V(G^*)| = |V(G)|$ and $|E(G^*)| = |E(G)|$. Since G is a τ-optimal graph, we have $t(G) \geq t(G^*)$. Since $k = k_1 + k_2 > k_2$, we may select a chain L_p from L_k in G^* with $p = k_2$, starting from u, and so

$$t(G^*) = t(G^* - L_p)p + t(G^*/L_p).$$

Note that $k - p = k_1$, which implies that $G^*/L_p = G/L_{k_2}$ and $(G^* - L_p)/L_q = (G - L_{k_1})/L_{k_2}$, where $L_q = L_k - L_p$. By Theorem 1, we have

$$t((G^* - L_p)/L_q) \leq t(G^* - L_p),$$

so

$$t(G^* - L_p) \geq t((G - L_{k_1})/L_{k_2}).$$

Therefore, we have

$$\begin{array}{rcl} t(G) & \geq & t(G^*) \\ & = & t(G^* - L_p)p + t(G^*/L_p) \\ & \geq & t((G - L_{k_1})/L_{k_2})p + t(G/L_{k_2}) \\ & > & t(G - L_{k_2})p + t(G/L_{k_2}) \\ & = & t(G), \end{array}$$

a contradiction.

PROOF of (b). We prove by contradiction. Let G be a τ-optimal graph, and assume that there are two chains L_{k_1} and L_{k_2} of G with

$$t(G - L_{k_1}) > t(G/L_{k_1}).$$

Let u and v be end vertices of L_{k_2}. We construct a new graph G^* from $G - L_{k_1}$ by adding a chain L_k in $(G-L_{k_1})/L_{k_2}$, with u, v as end vertices and $k = k_1$. Then we have $|V(G^*)| = |V(G)|$ and $|E(G^*)| = |E(G)|$. Since G is a τ-optimal graph, we have $t(G) \geq t(G^*)$ and $t(G^*) = t(G^* - L_k)k + t(G^*/L_k)$. Note that $G^*/L_k/L_{k_2} = (G - L_{k_1})/L_{k_2}$ and $G^* - L_k = G - L_{k_1}$. By Theorem 1, we have

$$t(G^*/L_k/L_{k_2}) \leq t(G^*/L_k),$$

so

$$t(G^*/L_k) \geq t((G - L_{k_1})/L_{k_2}).$$

Therefore, we have

$$\begin{array}{rcl} t(G) & \geq & t(G^*) \\ & = & t(G^* - L_k)k + t(G^*/L_k) \\ & \geq & t(G - L_{k_1})k_1 + t((G - L_{k_1})/L_{k_2}) \\ & > & t(G - L_{k_1})k_1 + t(G/L_{k_1}) \\ & = & t(G), \end{array}$$

a contradiction.

The following results may be useful.

LEMMA 1. [1] If $3 \leq n \leq e$, then τ-optimal graphs in $\Gamma(n, e)$ are two connected.

LEMMA 2. [1] Let G be a $\tau-$ optimal graph and $6 \leq n+2 \leq e$. If there exit two parallel chains L_{k_1}, L_{k_2} in G, then $k_1 = k_2 = 1$.

LEMMA 3. [4] Let G be a connected graph and $u, v \in V(G), d_G(u) = d_G(v) = 2$. If $u \notin N_G(v)$, then

$$t(G) \leq t(G/\{u, v\})$$

and the equality holds if and only if $N_G(u) = N_G(v)$, where $G/\{u, v\} = (G + uv)/uv$.

LEMMA 4. [1] Let G be a $\tau-$ optimal graph. If there exit two parallel chains L_{k_1}, L_{k_2} in G, then $|k_1 - k_2| \leq 1$.

LEMMA 5. If ε is an edge of G, then $t(G) = t(G - \varepsilon) + t(G/\varepsilon)$.

THEOREM 4. If $6 \leq n+2 < e, 1 < k < 3n - 2e + 2$, then

$$\widehat{t}(n, e) > 3\widehat{t}(n - k + 1, e - k),$$

where n, e, k are positive integer numbers and $\hat{t}(n, e)$ denotes the number of spanning trees of τ-optimal graphs in $\Gamma(n, e)$.

PROOF. Let $G' \in \Gamma(n - k + 1, e - k)$ be a $\tau-$ optimal graph. By Lemma 1, we know that G' is two connected. Since $1 < k < 3n - 2e + 2$, we obtain that the number of degree two vertices in G' is at least two. Without loss of generality, we assume $u, v \in V(G')$ and $|N_{G'}(u)| = |N_{G'}(v)| = 2$.

We distinguish two cases:

Case 1. $u \notin N_{G'}(v)$.

Since $6 \leq n - k + 3 \leq e - k$, by Lemma 2, we have $N_{G'}(u) \neq N_{G'}(v)$. We construct graph G as follows,

$$V(G) = V(G') \cup \{p_1, p_2, ..., p_{k-1}\},$$

$$E(G) = E(G') \cup \{(up_1), (p_1p_2), ..., (p_{k-1}v)\},$$

where u, v are two endpoints of L_k. Clearly $G \in \Gamma(n, e)$. By Lemma 3, we have

$$\begin{aligned}
\widehat{t}(n, e) &\geq t(G) \\
&= t(G - L_k)k + t(G/L_k) \\
&= t(G')k + t(G'/\{u, v\}) \\
&> 3t(G') \\
&= 3\hat{t}(n - k + 1, e - k).
\end{aligned}$$

Case 2. $u \in N_{G'}(v)$.

Without loss of generality, we assume that the number of degree two vertices in G' is two. Let $a \in N_{G'}(u), b \in N_{G'}(v)$. Since $4 + 3(n - k - 1) - 2e + 2k \geq 0$ and the equality holds if and only if $k = 3n - 2e + 1$, we have $d_{G'}(a) = d_{G'}(b) = 3$. By Lemma 4, we know $a \notin N(b)$.

Case 2.1. $N_{G'}(a) - u \neq N_{G'}(b) - v$.

Let $G'' = G' - \{u, v\} + (ab)$. By Lemma 3, we have

$$t(G'' - (ab)) = t(G' - \{u, v\}) < t((G' - \{u, v\})/\{a, b\}) = t(G''/(ab)).$$

We construct graph G as follows,

$$V(G) = V(G') \cup \{p_1, p_2, ..., p_{k-1}\},$$

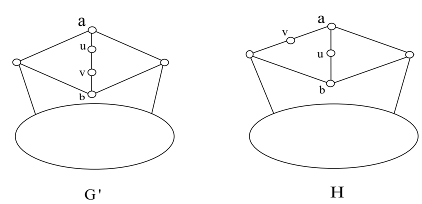
$$E(G) = E(G') \cup \{(up_1), (p_1p_2), ..., (p_{k-1}b)\},$$

where u, b are two endpoints of L_k. Clearly $G \in \Gamma(n, e)$. Analogously, we have

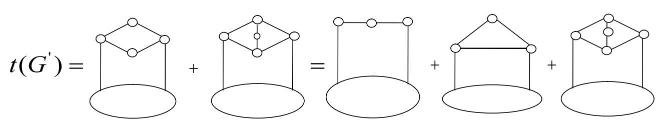
$$\begin{aligned}
\widehat{t}(n,e) &\geq t(G) \\
&= t(G - L_k)k + t(G/L_k) \\
&= t(G')k + 2t(G'') \\
&= t(G')k + 2t(G'' - (ab)) + 2t(G''/(ab)) \\
&> t(G')k + 3t(G'' - (ab)) + t(G''/(ab)) \\
&\geq 3t(G') \\
&= 3\hat{t}(n - k + 1, e - k).
\end{aligned}$$

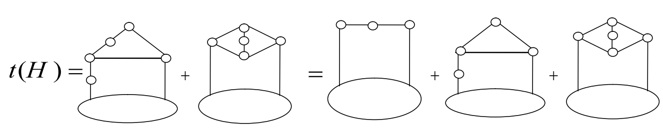
Case 2.2. $N_{G'}(a) - u = N_{G'}(b) - v$.

In this case, G' is the graph as follows.



By Lemma 5, we have





Clearly $t(H) > t(G')$, a contradiction. This means that this case is impossible.

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