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Undecidability Of Uzawa Equivalence Theorem And Cantor's Diagonal Argument*

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Abstract

The Uzawa equivalence theorem ([3]) showed that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. But the existence of an equilibrium price vector may be undecidable. We will show this undecidability using an extended version of Cantor's diagonal argument presented by Yanofsky [5] which is based on Lawvere [2].

1 Introduction

The existence of Walrasian equilibrium in an economy with continuous excess demand functions is proved by Brouwer's fixed point theorem. It is widely recognized that Brouwer's fixed point theorem is not a constructively proved theorem. The so-called Uzawa equivalence theorem ([3]) showed that the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. The existence of an equilibrium price vector, however, may be undecidable, and in [4] Velupillai said that the Uzawa equivalence theorem implies decidability of the halting problem of the Turing machine. In this paper we examine the decidability problem of the Uzawa equivalence theorem, properly speaking, the decidability problem of the existence of a Walrasian equilibrium price vector which is assumed in the Uzawa equivalence theorem, and show that the existence of an equilibrium price vector is not decidable using an extended version of Cantor's diagonal argument presented by Yanofsky [5] which is based on Lawvere [2].

According to [5] an extended version of Cantor's theorem is stated as follows.

> Let Y be a set, and $\alpha: Y \to Y$ a function without a fixed point (for all $y \in Y$, $\alpha(y) \neq y$), T and S sets and $\beta: T \to S$ a function that is onto (i.e., has a right inverse $\bar{\beta}: S \to T$), then for all functions $f: T \times S \to Y$ the function $g: T \to Y$ constructed as represented by the following diagram is not representable by f. Where Id is the identity mapping on T.

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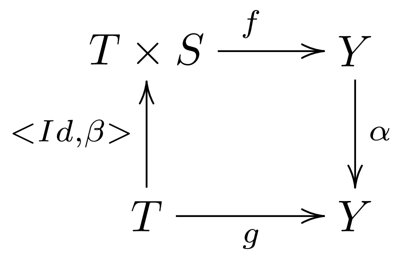


It is an extension of the famous Cantor's theorem that the cardinality of the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of positive integers is larger than the cardinality of $\mathbb{N}$, or in other words, there is no onto map $\mathbb{N} \to \mathcal{P}(\mathbb{N})$. In the proof of our main result we will use similar arguments to those in the proof of this theorem.

In the next section we present the theorem of the existence of Walrasian equilibrium and the Uzawa equivalence theorem. In Section 3 we present an extended version of Cantor's theorem and its proof, and prove the undecidability of the existence of an equilibrium price vector assumed in the Uzawa equivalence theorem.

2 Existence of Walrasian Equilibrium and Uzawa Equivalence Theorem

First we present the theorem of the existence of Walrasian equilibrium in an economy with continuous excess demand functions. Let Δ be an n-dimensional simplex ($n \geq 2$), and $p = (p_0, p_1, \cdots, p_n)$ be a point on Δ. $p_i \geq 0$ for each i and $\sum_{i=0}^{n} p_i = 1$. The prices of at least two goods are not zero. Thus, $p_i \neq 1$ for all i. Then, the theorem of the existence of Walrasian equilibrium is stated as follows.

THEOREM 1. (Existence of Walrasian equilibrium, for example, [1]) Consider an economy with $n+1$ goods X_0, X_1, ..., X_n with a price vector $p = (p_0, p_1, ..., p_n)$. Assume that an excess demand function for each good $f_i(p_0, p_1, .., p_n)$, $i = 0, 1, ..., n$, is continuous and satisfies the following condition.

$$p_0 f_0 + p_1 f_1 + \cdots + p_n f_n = 0 \text{ (Walras Law).}$$

Then, there exists an equilibrium price vector $(p_0^*, p_1^*, ..., p_n^*)$ which satisfies

$$f_i(p_0, p_1, ..., p_n) \leq 0$$

for all i ($i = 0, 1, ..., n$). And when $p_i > 0$ we have $f_i(p_0^*, p_1^*, ..., p_n^*) = 0$.

PROOF. See Appendix A.

Next we present the Uzawa equivalence theorem ([3]) which states that the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself.

THEOREM 2. (Uzawa equivalence theorem) The existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem.

PROOF. We will show the converse of the previous theorem. Let $\psi = \{\psi_0, \psi_1, \dots, \psi_n\}$ be an arbitrary continuous function from Δ to Δ, and construct excess demand functions as follows.

$$z_i(p) = \psi_i(p) - p_i\mu(p),\ \ i = 0, 1, \dots, n. \tag{1}$$

where $p = \{p_0, p_1, \dots, p_n\}$, and $\mu(p)$ is defined by

$$\mu(p) = \frac{\sum_{i=0}^{n} p_i\psi_i(p)}{\sum_{i=0}^{n} p_i^2}.$$

Each z_i for $i = 0, 1, \dots$ is continuous, and as we will show below, they satisfy the Walras Law. Let multiply p_i to each z_i in (1), and summing up them from 0 to n, we obtain

$$\sum_{i=0}^{n} p_i z_i = \sum_{i=0}^{n} p_i\psi_i(p) - \mu(p)\sum_{i=0}^{n} p_i^2 = \sum_{i=0}^{n} p_i\psi_i(p) - \frac{\sum_{i=0}^{n} p_i\psi_i(p)}{\sum_{i=0}^{n} p_i^2}\sum_{i=0}^{n} p_i^2$$
$$= \sum_{i=0}^{n} p_i\psi_i(p) - \sum_{i=0}^{n} p_i\psi_i(p) = 0.$$

Thus, z_i for all i satisfy the conditions of excess demand functions, and from Theorem 1 there exists an equilibrium price vector. Let $p^* = \{p_0^*, p_1^*, \dots, p_n^*\}$ be an equilibrium price vector. Then we have

$$\psi_i(p^*) \le \mu(p^*)p_i^*, \tag{2}$$

and if $p_i^* \neq 0$, $\psi_i(p^*) = \mu(p^*)p_i^*$. But since $\psi_i(p^*)$ must be non-negative by its definition (a function from Δ to Δ), we have $\psi_i(p^*) = 0$ when $p_i^* = 0$. Therefore, for all i we obtain $\psi_i(p^*) = \mu(p^*)p_i^*$. Summing up them from $i = 0$ to n, we get

$$\sum_{i=0}^{n} \psi_i(p^*) = \mu(p^*)\sum_{i=0}^{n} p_i^*.$$

Because $\sum_{i=0}^{n} \psi_i(p^*) = 1,\ \sum_{i=0}^{n} p_i^* = 1$, we have $\mu(p^*) = 1$, and so we obtain

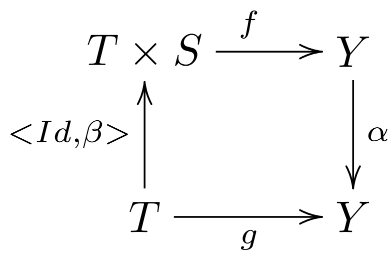
$$\psi_i(p^*) = p_i^*.$$

3 Uzawa Equivalence Theorem and an Extended Version of Cantor's Diagonal Argument

3.1 An extended Version of Cantor's Theorem

According to [5] we present an extended version of Cantor's theorem and its proof.

THEOREM 3. Let Y be a set, and $\alpha:\ Y \to Y$ a function without a fixed point (for all $y \in Y,\ \alpha(y) \neq y$), T and S sets and $\beta:\ T \to S$ a function that is onto (i.e., has a right inverse $\bar{\beta}: \ S \to T$), then for all functions $f: \ T \times S \to Y$ the function $g: \ T \to Y$ constructed as represented by the following diagram is not representable by f. Where Id is the identity mapping on T.



PROOF. See Appendix B.

The famous Cantor's theorem is derived as a corollary of this theorem.

THEOREM 4. The cardinality of the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is larger than the cardinality of $\mathbb{N}$, or in other words, there is no onto map $\mathbb{N} \to \mathcal{P}(\mathbb{N})$.

PROOF. See Appendix C.

3.2 Uzawa Equivalence Theorem and Cantor's Diagonal Argument

First in this section we show that for a real number b whether it satisfies $b \geq 0$ or $b \leq 0$ is not decidable. This undecidability means that for any real number b we can not decide $b \geq 0$ or $b \leq 0$ *in finite steps* by some procedure.

LEMMA 1. Whether a real number b satisfies $b \geq 0$ or $b \leq 0$ is not decidable. And this undecidability is proved using Cantor's diagonal argument.

PROOF. Consider a decimal expansion of b, $b = \sum_{i=1}^{M} d_i \times 10^{-i} \pm r \times 10^{-M-1}$, where all d_i are integers such that $0 \leq |d_i| \leq 9$ and all $d_i \geq 0$ or all $d_i \leq 0$, M is a sufficiently large positive integer, and a positive integer r ($0 \leq r \leq 9$) is an error bound. Let X be the set of decimal expansions of real numbers, $\mathbb{N}$ be the set of positive integers, Id be the identity mapping on $X \times \mathbb{N}$, and $\beta: \ X \times \mathbb{N} \to X$ is a projection function, that is, $\beta(b, N) = b$. $X \times \mathbb{N}$ is the product set of X and $\mathbb{N}$. Assume that there exists some procedure by which we can decide whether (a decimal expansion of) *every* real number $x \in X$ satisfies $x \geq 0$ or $x \leq 0$ in *finite steps*. We decide $x \geq 0$ or $x \leq 0$ reading each digit of a decimal expansion of x step by step in at most $N \in \mathbb{N}$ steps, where $N \leq M$, that is, within N digits of the decimal expansion of x. Let b and x be two real numbers and $Y = \{-10^{-N}, 10^{-N}\}$. Denote the output of this procedure by a function $f: \ X \times \mathbb{N} \times X \to Y$. f is defined as follows.

$$\begin{cases} f(b, N, x) = 10^{-N} \text{ if we decide } x \geq 0. \\ f(b, N, x) = -10^{-N} \text{ if we decide } x \leq 0. \end{cases}$$

Define α as follows.

$$\alpha: \ Y \to Y: \ \alpha(10^{-N}) = -10^{-N} \text{ and } \alpha(-10^{-N}) = 10^{-N}.$$

We construct $g: \ X \times \mathbb{N} \to Y$ which determines the value of a real number b, that is, $g(b, N) = b$ as the following composition of three functions, $< Id, \beta >$, f and α.

$$\begin{array}{ccc} (X \times \mathbb{N}) \times X & \xrightarrow{f} & Y \\ {\scriptstyle <Id,\beta>} \uparrow & & \downarrow {\scriptstyle \alpha} \\ (X \times \mathbb{N}) & \xrightarrow[g]{} & Y \end{array}$$

$$g(b, N) = b = \alpha(f(b, N, \beta(b, N))) = \alpha(f(b, N, b)).$$

Since $g(b, N) = b$ is itself a real number, $f(x, N, b)$ must be 10^{-N} when $g(b, N) = 10^{-N}$ and $f(x, N, b)$ must be -10^{-N} when $g(b, N) = -10^{-N}$ for all x. Thus, $g(-, N)$ must be representable by $f(x, N, -)$. But, since α has no fixed point, by Theorem 3 $g(-, N)$ is not representable by $f(x, N, -)$. Therefore, there does not exist any procedure to decide whether any real number b satisfies $b \geq 0$ or $b \leq 0$ in finite steps.

For all i other than 0 ψ_i is assumed to be defined as follows

$$\psi_i = \frac{\lambda_i(p_i)}{\sum_{j=0}^n \lambda_j(p_j)}.$$

And for $i = 0$ we assume

$$\psi_0 = \frac{\lambda_0(p_0)}{\sum_{j=0}^n \lambda_j(p_j)}.$$

Then we have

$$z_i(p) = \frac{\lambda_i(p_i)}{\sum_{j=0}^n \lambda_j(p_j)} - p_i \frac{\sum_{j=0}^n p_j \lambda_j(p_j)}{\sum_{j=0}^n p_j^2 \sum_{j=0}^n \lambda_j(p_j)}, \text{for all } i \neq 0,$$

and

$$z_0(p) = \frac{\lambda_0(p_0)}{\sum_{j=0}^n \lambda_j(p_j)} - p_0 \frac{\sum_{j=0}^n p_j \lambda_j(p_j)}{\sum_{j=0}^n p_j^2 \sum_{j=0}^n \lambda_j(p_j)}.$$

If $z_i = 0$ for all i including $i = 0$, then we obtain

$$p_0 \lambda_i(p_i) = p_i \lambda_0(p_0), \text{for all } i \neq 0. \tag{3}$$

Now specifically we assume

$$\lambda_i(p_i) = p_i + 1, \ i \neq 0, \tag{4}$$

and

$$\lambda_0(p_0) = \begin{cases} \frac{np_0}{1-p_0} + \frac{1}{4} + b, \text{ when } p_0 < \frac{1}{4} \\ \frac{np_0}{1-p_0} + p_0 + b, \text{ when } \frac{1}{4} \leq p_0 \leq \frac{1}{2} \\ \frac{np_0}{1-p_0} + \frac{1}{2} + b, \text{ when } \frac{1}{2} < p_0 < 1 \end{cases} \tag{5}$$

where b is a real number such that $b > -\frac{1}{4}$. From (3) and (4) we have

$$p_i(\lambda_0(p_0) - p_0) = p_0, \ i \neq 0. \tag{6}$$

This implies that all $p_i, \ i \neq 0$, are equal. Since $\sum_{j=0}^{n} p_j = np_i + p_0 = 1$ we have

$$p_i = \frac{1 - p_0}{n}. \tag{7}$$

If $p_0 = 0$, we have $p_i = \frac{1}{n}$ for all $i \neq 0$. But, then since $\lambda_0(p_0) = \frac{1}{4} + b > 0$ it contradicts (6). Thus $p_0 \neq 0$. From (6) and (7)

$$(1 - p_0)(\lambda_0(p_0) - p_0) = np_0. \tag{8}$$

Therefore, from (5) and (8) we obtain

$$\begin{cases} p_0 - \frac{1}{4} - b = 0, \text{ when } p_0 < \frac{1}{4} \\ b = 0, \text{ when } \frac{1}{4} \leq p_0 \leq \frac{1}{2} \\ p_0 - \frac{1}{2} - b = 0, \text{ when } p_0 > \frac{1}{2} \end{cases} \tag{9}$$

These are the equilibrium conditions. The assumption of the existence of Walrasian equilibrium implies the existence of p_0 in $(0, 1)$ such that one of these conditions is satisfied. Which of the conditions is satisfied depends on the value of b.

Now we show the following result.

LEMMA 2. The existence of an equilibrium price vector implies that for a real number b we can decide $b \geq 0$ or $b \leq 0$.

PROOF. Let p_0^* be an equilibrium value of p_0. If $b < 0$, we have $p_0^* < \frac{1}{4}$. If $b = 0$, p_0^* is any value in $\left[\frac{1}{4}, \frac{1}{2}\right]$. On the other hand, if $b > 0$, we have $p_0^* > \frac{1}{2}$. About three real numbers p_0^*, $\frac{1}{4}$ and $\frac{1}{2}$ we have $p_0^* > \frac{1}{4}$ or $p_0^* < \frac{1}{2}$.

> Consider a decimal expansion of p_0^*, $p_0^* = \sum_{i=1}^{M} a_i \times 10^{-i} \pm r \times 10^{-M-1}$, where all a_i are non-negative integers such that $0 \leq a_i \leq 9$, M is a sufficiently large positive integer, and a positive integer r $(0 \leq r \leq 9)$ is an error bound. If $a_1 \leq 3$, we find $p_0^* < \frac{1}{2}$, on the other hand if $a_1 \geq 3$, we find $p_0^* > \frac{1}{4}$. Therefore, we can decide $p_0^* > \frac{1}{4}$ or $p_0^* < \frac{1}{2}$ in one step.

If $p_0^* > \frac{1}{4}$, then b must satisfy $b \geq 0$. And if $p_0^* < \frac{1}{2}$, then b must satisfy $b \leq 0$. Therefore, in order to determine an equilibrium price p_0^* we must know whether $b \geq 0$ or $b \leq 0$.

From Lemma 1 and 2 we obtain the main result of this paper.

THEOREM 5. The existence of an equilibrium price vector assumed in the Uzawa equivalence theorem is undecidable.

4 Final Remark

The Uzawa equivalence theorem in general equilibrium theory demonstrates that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem. We have shown that the existence of a Walrasian equilibrium price vector assumed in the Uzawa's theorem is undecidable using an extended version of Cantor's diagonal argument.

Appendices

A Proof of Theorem 1

Let v_i be a function from $p = (p_0, p_1, ..., p_n)$ to $v = (v_0, v_1, ..., v_n)$ as follows,

$$v_i = p_i + f_i,\ \text{when}\ f_i > 0,$$

$$v_i = p_i,\ \text{when}\ f_i \leq 0.$$

We construct a function $\varphi = (\varphi_0, \varphi_1, ..., \varphi_n)$ from Δ to Δ as follows.

$$\varphi_i(p_0, p_1, ..., p_n) = \frac{1}{v_0 + v_1 + \cdots + v_n} v_i.$$

Since we have $\varphi_i \geq 0,\ i = 0, 1, ..., n$, and

$$\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1.$$

$(\varphi_0, \varphi_1, ..., \varphi_n)$ is a point on Δ.

Since each f_i is continuous, each φ_i is also continuous. Thus, by Brouwer's fixed point theorem there exists $p^* = (p_0^*, p_1^*, ..., p_n^*)$ that satisfies

$$(\varphi_0(p_0^*, p_1^*, ..., p_n^*), \varphi_1(p_0^*, p_1^*, ..., p_n^*), ..., \varphi_n(p_0^*, p_1^*, ..., p_n^*)) = (p_0^*, p_1^*, ..., p_n^*).$$

Since $v_i \geq p_i$ for all i, we have $v_i(p_0^*, p_1^*, ..., p_n^*) = \lambda p_i^*$ for all i for some $\lambda \geq 1$. We will show $\lambda = 1$. Now assume $\lambda > 1$. Then, if $p_i^* > 0$ we have $v_i(p_0^*, p_1^*, ..., p_n^*) > p_i^*$, that is, $f_i(p_0^*, p_1^*, ..., p_n^*) > 0$. On the other hand, since for all i $p_i^* \geq 0$ and the sum of them is one, at least one of them is positive. Then, we have $p_0^* f_0 + p_1^* f_1 + \cdots + p_n^* f_n > 0$. It contradicts the Walras Law. Therefore, we get $\lambda = 1$. And we obtain $v_0 = p_0^*,\ v_1 = p_1^*, ...,\ \ v_n = p_n^*$ and $f_i(p_0^*, p_1^*, ..., p_n^*) \leq 0$ for all i.

B Proof of Theorem 3

Let Y, α, T and β be given. Let $\bar{\beta}:\ S \to T$ be the right inverse of β. By definition

$$g(t) = \alpha(f(t, \beta(t))).$$

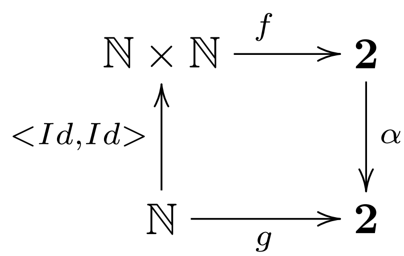
We show that for all $s \in S$ $g(-) \neq f(-, s)$. If $g(-) = f(-, s)$, then evaluation at $\bar{\beta}(s_0)$ gives

$$f(\bar{\beta}(s_0), s_0) = g(\bar{\beta}(s_0))\ = \alpha(f(\bar{\beta}(s_0), \beta(\bar{\beta}(s_0))))\ = \alpha(f(\bar{\beta}(s_0), s_0)\ .$$

where the first equality follows from the representability of g, the second from the definition of g and the third from the definition of right inverse. This means that α has a fixed point $f(\bar{\beta}(s_0), s_0)$. It is a contradiction.

C Proof of Theorem 4

For T, S, β and α in Theorem 3 we assume that $T = S = \mathbb{N}$, $\beta = Id$ be the identity mapping on T, $Y = \mathbf{2} = \{0, 1\}$ and α be a function such that $\alpha(1) = 0$ and $\alpha(0) = 1$. Assume that there is an onto map $h : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, and denote $h(n) = S_{h(n)}$. $S_{h(n)}$ is a subset of $\mathbb{N}$. Consider a function $f :\ \mathbf{N} \times \mathbf{N} \to \mathbf{2}$ such that $f(n, m) = 1$ when $n \in S_{h(m)}$ and $f(n, m) = 0$ when $n \notin S_{h(m)}$ for $n,\ m \in \mathbb{N}$. A function $g :\ \mathbb{N} \to \mathbf{2}$ is constructed as represented by the following diagram



$$g(n) = \alpha(f(n, n)).$$

$g(n)$ is a characteristic function of the set

$$G = \{n|\ n \notin S_{h(n)}\}.$$

$g(n) = 1$ when $n \in G$ and $g(n) = 0$ when $n \notin G$. Since G is a subset of $\mathbb{N}$, we have $g(-) = f(-, m)$ for some $m \in \mathbb{N}$, that is, $g(-)$ must be representable by $f(-, m)$. But, since α has no fixed point, by Theorem 3 $g(-)$ is not representable by $f(-, m)$. Therefore, there does not exist an onto map $h : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

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