Alternate Derivations Of The Stability Region Of A Difference Equation With Two Delays[*]

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Abstract

The asymptotic stability of the difference equation $f_k = af_{k-m} + bf_{k-n}$, where a, b are real numbers and n and m are fixed positive integers, has been examined by several authors recently, and stability conditions are derived by studying its associated characteristic polynomial. In this paper, we provide an alternate but elementary approach to this problem and hope that our method will lead us to new tools for dealing with stability of other difference equations.

1 Introduction

The asymptotic stability of the difference equation

$$f_k = af_{k-m} + bf_{k-n}, \ k = n, n+1, ...,\tag{1}$$

where a, b are real numbers and n and m are positive integers, has been considered by several authors recently (see e.g. [1-16] in which reasons for studying such a problem are also provided). In particular, in Dannan [12] and in Kipnis and Nigmatullin [15,16], necessary and sufficient conditions for the asymptotic stability are asserted. Unfortunately, as pointed out by Ren in [13], Dannan's results are based on a statement (Lemma 6 in [1]: If $f(t) = \sin mt / \sin nt$ where m, n are positive integers such that $\sin nt \neq 0$, then $f(t)f'(t) > 0$ for $m < n$ and $f(t)f'(t) < 0$ for $m > n$), which is wrong (e.g. by considering the function $f(t) = \sin 2t / \sin t$, or $\sin t / \sin 2t$). In [16], the stability conditions are correct and their derivations are based on the principle of arguments applied to the associated characteristic polynomial

$$P(\lambda|a, b) \equiv \lambda^n - a\lambda^{n-m} - b\tag{2}$$

together with tedious analysis of the winding numbers of the hodograph of $P(e^{i\omega}|a, b)$.

In view of the importance of equation (1) and other similar equations, it is of interest to approach the same problem by different means. In this paper, we will obtain the same stability conditions[1], but our proofs will be based on considering the properties of the parametric functions $a = a(r, \omega)$, $b = b(r, \omega)$ solved from $P(re^{i\omega}|a, b) = 0$.

This approach uses the continuity of the maximal magnitude of roots of polynomials with respect to their coefficients and other elementary properties of the sine and cosine functions and hence is accessible to the audience not equipped with the tools in complex analysis. Furthermore, since our approach is different, we will be able to obtain additional information on the stability regions not provided in [15,16].

To this end, let us first mention the fact that our stability question is easily transformed to when does the characteristic polynomial has only subnormal roots (a root is subnormal if its modulus is less than 1). Since $P(\lambda|a,b) = 0$ can be written as

$$a\lambda^{-m} + b\lambda^{-n} = 1, \tag{3}$$

we may assume without loss of generality that $1 \leq m \leq n$. If $n = m$, then (3) can be written as

$$\lambda^m = a + b,$$

thus its roots are subnormal if, and only if, $|a + b| < 1$. Furthermore, if n and m have a common factor μ, then $m = \mu\tau, n = \mu\sigma$ and

$$P(\lambda|a,b) = (\lambda^\mu)^\sigma - a(\lambda^\mu)^{\sigma-\tau} - b = 0. \tag{4}$$

Since every root of (4) is subnormal if, and only if, every root of the following equation

$$\xi^\sigma - a\xi^{\sigma-\tau} - b = 0$$

is subnormal, we may assume further that m and n do not have any common factors other than one. For these reasons, we will assume throughout the rest of our investigations that

(H1) the positive integers n and m do not have any common factors other than 1, and
         $1 \leq m < n$.

For each pair (n, m) that satisfies (H1), we will determine the set $\Omega(n, m)$ of real number pairs of the form (a, b) such that every root of (2) is subnormal. The set $\Omega(n, m)$ is a subset of the x, y-plane and is naturally called the region of (asymptotic) stability of (1).

The parity of the integers n and m will play important roles in the sequel. For this reason, we will find the stability regions for the following mutually exclusive and exhaustive cases:

(H2) n is even, m is odd;

(H3) n is odd, m is odd;

(H4) n is odd, m is even.

For the sake of convenience, the maximum of the absolute values of the roots of (2) is denoted by

$$\rho(a, b) = \max\{|\lambda| : P(\lambda|a, b) = 0\}.$$

It is well known that, for fixed n and m, $\rho(a, b)$ (as the spectral radius of a real matrix) is a continuous function with respect to (a, b).

2 Partitions

We say that $\{x_0, x_1, ..., x_n\}$ is a partition of the closed interval $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. If $x_{i+1} - x_i = (b - a)/n$ for any $i \in \{0, 1, ..., n - 1\}$, the partition $\{x_0, x_1, ..., x_n\}$ is then said to be uniform.

Let $P = \{p_0, p_1, ..., p_n\}$, $Q = \{q_0, q_1, ..., q_m\}$ and $W = \{w_0, w_1, ..., w_{n-m}\}$ be uniform partitions of $[0, \pi]$; and let $P' = \{p'_0, p'_1, ..., p'_{2n}\}$, $Q' = \{q'_0, q'_1, ..., q'_{2m}\}$ and $W' = \{w'_0, w'_1, ..., w'_{2(n-m)}\}$ be uniform partitions of $[0, 2\pi]$. Clearly, $p'_i = p_i = \frac{i\pi}{n}$ for $i \in \{0, 1, ..., n\}$, $q'_j = q_j = \frac{j\pi}{m}$ for $j \in \{0, 1, ..., m\}$ and $w'_v = w_v$ for $v \in \{0, 1, ..., n - m\}$.

We will also let

$$I_v = (w'_v, w'_{v+1}) \text{ for } v \in \{0, 1, ..., 2(n - m) - 1\}.$$

Note that for $v \in \{0, 1, ..., n - m - 1\}$, the interval I_v can also be written as (w_v, w_{v+1}).

The reason for considering the partitions P, Q and W is that the set of roots of $\sin n\theta = 0$ in $[0, \pi]$ is P, the set of roots of $\sin m\theta = 0$ in $[0, \pi]$ is Q, and the set of roots of $\sin(n - m)\theta = 0$ in $[0, \pi]$ is W. These facts will be useful when we consider the parametric function defined by (13) in a later section.

The partitions P, Q and W are clearly related. For example, let $n = 8$ and $m = 5$, then we may easily check that

$$
\begin{aligned}
0 &= p_0 = q_0 = w_0, \\
w_3 &= q_5 = p_8 = \pi, \\
I_0 &= (w_0, w_1) = (0, \pi/3), \\
I_1 &= (w_1, w_2) = (\pi/3, 2\pi/3), \\
I_2 &= (w_2, w_3) = (2\pi/3, \pi),
\end{aligned}
$$

and

$$w_0 < p_1 < q_1 < p_2 < w_1 < p_3 < q_2 < p_4 < q_3 < p_5 < w_2 < p_6 < q_4 < p_7 < w_3. \quad (5)$$

As another example, let $n = 8$ and $m = 1$, then

$$
\begin{aligned}
0 &= p_0 = q_0 = w_0, \\
w_7 &= p_8 = q_1 = \pi,
\end{aligned}
$$

and

$$w_0 < p_1 < w_1 < p_2 < w_2 < p_3 < w_3 < p_4 < w_4 < p_5 < w_5 < p_6 < w_6 < p_7 < w_7.$$

LEMMA 1. Suppose (H1) holds. Then
(1) For any i in $\{1, 2, ..., n - 2\}$, $q_j \in (p_i, p_{i+1})$ for some $j \in \{1, 2, ..., m - 1\}$, or $w_v \in (p_i, p_{i+1})$ for some $v \in \{1, 2, ..., n - m - 1\}$; and
(2) $w_v \notin (p_0, p_1) \cup (p_{n-1}, p_n)$ for any $v \in \{0, 1, ..., n - m\}$, and, $q_j \notin (p_0, p_1) \cup (p_{n-1}, p_n)$ for any $j \in \{0, 1, ..., m\}$.

PROOF. Let f be the continuous function

$$f(t) = \sin(mt) \sin(nt - mt), \ t \in [0, \pi]. \quad (6)$$

For any $i \in \{1, 2, ..., n-1\}$, since

$$f(p_i) f(p_{i+1}) = \sin(mp_i) \sin(i\pi - mp_i) \sin(mp_{i+1}) \sin((i+1)\pi - mp_{i+1}) < 0,$$

by the intermediate value theorem, we have $f(x^*) = 0$, that is, $\sin(mx^*) = 0$ or $\sin(n-m)x^* = 0$ for some $x^* \in (p_i, p_{i+1})$. Since $i \in \{1, 2, ..., n-2\}$, we see that 0 and π are not in the interval (p_i, p_{i+1}). Hence $x^* = q_j$ for some $j \in \{1, 2, ...m-1\}$ or $x^* = w_v$ for some $v \in \{1, 2, ...n-m-1\}$. The proof of our first assertion is complete.

Since

$$p_1 - p_0 = p_n - p_{n-1} = \frac{\pi}{n} < \frac{\pi}{n-m} = w_v - w_{v-1}$$

for all $v \in \{1, 2, ..., n-m\}$, we see that $w_v \notin (p_0, p_1) \cup (p_{n-1}, p_n)$ for all $v \in \{0, 1, ..., n-m\}$. Similarly, $q_j \notin (p_0, p_1) \cup (p_{n-1}, p_n)$ for all $j \in \{0, 1, ..., m\}$. The proof is complete.

LEMMA 2. Suppose (H1) holds. Then

(1) $W \cap P = \{0, \pi\}$, $W \cap Q = \{0, \pi\}$, $P \cap Q = \{0, \pi\}$, $W' \cap P' = \{0, \pi, 2\pi\}$, $W' \cap Q' = \{0, \pi, 2\pi\}$ and $P' \cap Q' = \{0, \pi, 2\pi\}$.

(2) For any (w_v, w_{v+1}) where $v \in \{0, 1, ..., n-m-1\}$, there are $p_{i+1}, p_{i+2}, ..., p_{i+k}$ in P such that $w_v < p_{i+1} < p_{i+2} < ... < p_{i+k} < w_{v+1}$ and $p_i \leq w_v$, $p_{i+k+1} \geq w_{v+1}$. Furthermore, when $k = 1$, $(w_v, w_{v+1}) \cap Q = \emptyset$ and when $k \geq 2$, there are $q_{j+1}, q_{j+2}, ..., q_{j+k-1}$ in Q such that

$$w_v < p_{i+1} < q_{j+1} < p_{i+2} < ... < q_{j+k-1} < p_{i+k} < w_{v+1}, \tag{7}$$

and $q_j \leq w_v$ as well as $q_{j+k} \geq w_{v+1}$.

PROOF. Clearly, 0 and π are inside $W \cap P$, $W \cap Q$ and $Q \cap P$. Suppose $(W \cap P) \backslash \{0, \pi\} \neq \emptyset$. Then $w_v = p_i$ for some $v \in \{1, 2, ..., n-m-1\}$ and $i \in \{1, 2, ..., n-1\}$. So $\frac{v}{i} = \frac{n-m}{n}$. But $\gcd(n, n-m) = 1$, $i < n$ and $v < n-m$ are contradictory statements. Hence $W \cap P = \{0, \pi\}$. By similar arguments, we see that the rest of the assertions in (1) hold.

To show (2), let x_v be the number of elements of the set $P \cap I_v$ and y_v be the number of the element of the set $Q \cap I_v$. Since $\pi/(n-m) > \pi/n$, we see that $x_v > 0$ for $v \in \{0, 1..., n-m-1\}$. By Lemma 1, $x_v - 1 \leq y_v$ for $v \in \{0, 1..., n-m-1\}$. We assert that $x_v - 1 = y_v$. Suppose to the contrary that $x_v - 1 < y_v$ for some v. Then

$$m - 1 = \sum_{v=0}^{n-m-1} y_v > \sum_{v=0}^{n-m-1} (x_v - 1) = n - 1 - (n - m) = m - 1,$$

which is a contradiction. Hence $x_v - 1 = y_v$ for all v. By Lemma 1, the condition (7) holds. The proof is complete.

By Lemma 1 and Lemma 2, $P \cap Q \cap W = \{0, \pi\}$, the set $(w_v, w_{v+1}) \cap P$ is not empty for any $v \in \{0, 1, ..., n-m-1\}$, $(q_j, q_{j+1}) \cap P$ is not empty for any $j \in \{0, 1, ..., m-1\}$, and one of the sets $(p_i, p_{i+1}) \cap W$ and $(p_i, p_{i+1}) \cap Q$ is also nonempty for any $i \in \{0, 1, ..., n-1\}$. Therefore $P \cup Q \cup W$ is a partition of $[0, \pi]$ and is of the form $\{\xi_0, \xi_1, ..., \xi_{2n-2}\}$; furthermore, each interval (ξ_k, ξ_{k+1}), where $k \in \{0, 1, ..., 2n-3\}$, must satisfy one of the following four conditions:

(C1) $(\xi_k, \xi_{k+1}) = (w_v, p_i)$ for some $w_v \in W$ and $p_i \in P \backslash \{\pi\}$.

(C2) $(\xi_k, \xi_{k+1}) = (p_i, q_j)$ for some $p_i \in P$ and $q_j \in Q\backslash\{\pi\}$.

(C3) $(\xi_k, \xi_{k+1}) = (q_j, p_i)$ for some $q_j \in Q\backslash\{0\}$ and $p_i \in P\backslash\{\pi\}$.

(C4) $(\xi_k, \xi_{k+1}) = (p_i, w_v)$ for some $p_i \in P\backslash\{0\}$ and $w_v \in W$.

For example, in view of (5), we see that for $n = 8$ and $n = 5$,

$$
\begin{aligned}
\xi_0 &= w_0 < \xi_1 = p_1 < \xi_2 = q_1 < \xi_3 = p_2, \\
p_2 &< \xi_4 = w_1 < \xi_5 = p_3 < \xi_6 = q_2 < \xi_7 = p_4 < \xi_8 = q_3, \\
q_3 &< \xi_9 = p_5 < \xi_{10} = w_2 < \xi_{11} = p_6 < \xi_{12} = q_4 < \xi_{13} = p_7 < \xi_{14} = w_3.
\end{aligned}
$$

3 Normal Root Curves

We first find necessary and/or sufficient conditions for a and b such that $P(\lambda|a, b)$ has certain types of roots: we will treat the pair (a, b) as a point in the x, y-plane, then we will see that it lies on certain curves in the plane.

For this purpose, we consider roots of the form $re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi]$. If $re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi]$, is a root of $P(\lambda|a, b)$, then

$$
\begin{aligned}
r^n \cos(n\theta) - ar^{n-m}\cos(n-m)\theta - b &= 0, & (8) \\
r^n \sin(n\theta) - ar^{n-m}\sin(n-m)\theta &= 0. & (9)
\end{aligned}
$$

The system (8)-(9) can be treated as a pair of linear equations in a and b. Let us therefore consider the coefficient matrix

$$
A(r, \theta) = \begin{pmatrix} r^{n-m}\cos(n\theta - m\theta) & 1 \\ r^{n-m}\sin(n\theta - m\theta) & 0 \end{pmatrix}, \ r > 0, \theta \in [0, 2\pi].
$$

Note that $\det A(r, \theta) = -r^{n-m}\sin(n\theta - m\theta) = 0$ if, and only if, $\theta \in W'$. If $\det A(r, \theta) = 0$, then by (9), $\sin(n\theta) = 0$ as well. So $\theta \in W' \cap P'$. Thus, $\theta \in \{0, \pi, 2\pi\}$ by Lemma 2. If $\theta = 0$ or 2π, then (8) and (9) can be written as

$$
r^n - ar^{n-m} - b = 0,
$$

and if $\theta = \pi$, then (8) and (9) can be written as

$$
r^n \cos(n\pi) - ar^{n-m}\cos(n-m)\pi - b = 0.
$$

Let

$$
L_1^{(r)} = \{(x, y) \in R^2 : r^n - xr^{n-m} - y = 0\}, \ r > 0 \tag{10}
$$

and

$$
L_2^{(r)} = \{(x, y) \in R^2 : r^n \cos(n\pi) - xr^{n-m}\cos(n-m)\pi - y = 0\}, \ r > 0. \tag{11}
$$

We see that if $\theta = 0$ or 2π, then $(a, b) \in L_1^{(r)}$; while if $\theta = \pi$, then $(a, b) \in L_2^{(r)}$.

On the other hand, suppose $\det A(r, \theta) \neq 0$. Then from (8) and (9), we may solve for a and b:

$$
a = \frac{\sin(n\theta)}{\sin(n-m)\theta}r^m, \ b = \frac{-\sin(m\theta)}{\sin(n-m)\theta}r^n, \ \theta \in [0, 2\pi]\backslash W', \tag{12}
$$

where we recall that $I_v = (w'_v, w'_{v+1})$ and $\left\{ w'_0, w'_1, ..., w'_{2(n-m)-1} \right\}$ is a partition of $[0, 2\pi]$. If we let $(x_r(\theta), y_r(\theta))$ be the parametric function defined by

$$x_r(\theta) = \frac{\sin(n\theta)}{\sin(n-m)\theta} r^m, \ y_r(\theta) = \frac{-\sin(m\theta)}{\sin(n-m)\theta} r^n, \ \theta \in [0, 2\pi]\backslash W', \tag{13}$$

then the above condition is equivalent to (a, b) lying on one of the curves

$$S_v^{(r)} = \left\{ (x_r(\theta), y_r(\theta)) : x_r(\theta) = \frac{\sin(n\theta)}{\sin(n-m)\theta} r^m, \ y_r(\theta) = \frac{-\sin(m\theta)}{\sin(n-m)\theta} r^n, \ \theta \in I_v \right\}, \tag{14}$$

where $v \in \{0, 1, ..., 2(n-m) - 1\}$. Fix $r > 0$. Since for $t \in (0, \pi)$,

$$x_r(\pi + t) = x_r(\pi - t) \text{ and } y_r(\pi + t) = y_r(\pi - t),$$

hence $S_0^{(r)} = S_{2(n-m)-1}^{(r)}$, $S_1^{(r)} = S_{2(n-m)-2}^{(r)}$, ..., so that the condition (12) is equivalent to (a, b) lying on one of the curves $S_v^{(r)}$ where $v = 0, 1, ..., n - m - 1$.

We summarize the above discussions as follows.

LEMMA 3. Suppose (H1) holds. If $re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi]$, is a root of $P(\lambda | a, b) = \lambda^n - a\lambda^{n-m} - b$, then (a, b) lies on the curves $L_1^{(r)}$, $L_2^{(r)}$, $S_0^{(r)}$, $S_1^{(r)}$, ..., $S_{n-m-2}^{(r)}$ or $S_{n-m-1}^{(r)}$.
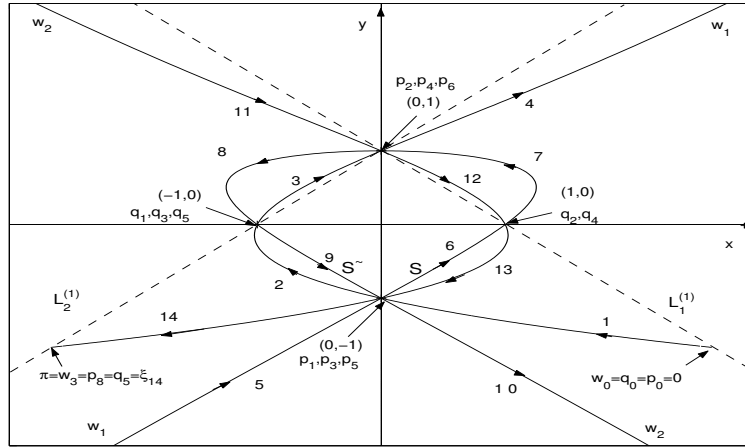

Figure 1: $n = 8, m = 5$

As an example, consider the case where $n = 8$ and $m = 5$. For $r = 1$, the function $(x_1(\theta), y_1(\theta))$ from $\theta = 0^+$ to $\theta = \pi^-$ is traced and the resulting (directed) curves are depicted in Figure 1.

Roughly, we see that $S_0^{(1)}$ over the interval (w_0, w_1) is composed of directed segments marked by 1, 2, 3 and 4; $S_1^{(1)}$ over the interval (w_1, w_2) is composed of directed segments

marked by $5, 6, 7, 8, 9$ and 10; and $S_2^{(1)}$ over the interval (w_2, w_3) is composed of directed segments marked by $11, 12, 13$ and 14. Note that as we traced the curve $S_0^{(1)}$, we pass through the point $(0, -1)$ which takes place at $\theta = p_1$, then the point $(-1, 0)$ at $\theta = q_1$, and then the point $(0, 1)$ at $\theta = p_2$. Similar assertions can also be made for the other two curves $S_1^{(1)}$ and $S_2^{(1)}$. Such assertions should not be surprising since $\sin(np_i) = 0$ for $i = 0, ..., 8$, and $\sin(mq_j) = 0$ for $j = 0, ..., 5$. Furthermore, we see that all the p_i with even i are placed by the side of $(0, 1)$, while those with odd i by the side of $(0, -1)$; and all the q_j with even j by the side of $(1, 0)$ and those with odd j by the side of $(-1, 0)$.

The above example provides clues to several general facts. First of all, it is easy to see that for $i \in \{1, 2, ..., n-1\}$, $(x_1(p_i), y_1(p_i)) = (0, -1)$ if i is odd and $(x_1(p_i), y_1(p_i)) = (0, +1)$ if i is even; and for $j \in \{1, ..., m-1\}$, $(x_1(q_j), y_1(q_j)) = (1, 0)$ if j is even and $(x_1(q_j), y_1(q_j)) = (-1, 0)$ if j is odd. Next, we consider the location of the root curves. This is accomplished by considering the roots of $P(\lambda|a, b)$ when (a, b) lies in one of the four open quadrants. We will discuss the case where $a > 0$ and $b < 0$, the other three cases being similar. Let $a > 0$ and $b < 0$. Suppose $re^{i\theta}$, where $r \geq 0$ and $\theta \in [0, \pi]$, is a root of the polynomial $P(\lambda|a, b)$. Since $P(0|a, b) = b \neq 0$, we see that $r > 0$. Next, we consider four mutually disjoint and exhaustive cases for θ: (A) $\theta \in W$; (B) $\theta \in P \backslash W$; (C) $\theta \in Q \backslash W$; and (D) $\theta \in [0, \pi] \backslash \{\xi_0, \xi_1, ..., \xi_{2n-2}\}$.

(A) If $\theta \in W$, then $\sin(n - m)\theta = 0$, and thus by (9), $\sin(n\theta) = 0$ as well. So $\theta \in P$. By Lemma 2, $\theta = 0$ or π.

(B) If $\theta \in P \backslash W$, then by (12), $a = x^{(r)}(\theta) = 0$, which is contrary to $a > 0$.

(C) If $\theta \in Q \backslash W$, then by (12), $b = y^{(r)}(\theta) = 0$, which is contrary to $b < 0$.

(D) If $\theta \in [0, \pi] \backslash \{\xi_0, \xi_1, ..., \xi_{2n-2}\}$, then $\theta \in (\xi_k, \xi_{k+1})$ for some $k \in \{0, 1, ..., 2n - 3\}$. Hence we need to consider the following subcases (C1)-(C4) in the previous section:

(a) In case (C1), $\theta \in (\xi_k, \xi_{k+1}) = (w_v, p_i)$ for some $w_v \in W$ and $p_i \in P \backslash \{p_n\}$. By Lemma 2, $q_j \leq p_{i-1} \leq w_v < p_i < q_{j+1}$ for some $q_j, q_{j+1} \in Q$ and $p_{i-1} \in P$. Since $a > 0$, $b < 0$ and $\sin(n\theta)$, $\sin(m\theta)$ and $\sin(n - m)\theta$ have the same sign by (12), we see that if i is even, then $j < m$ and v are odd. Since

$$(x_r(p_i), y_r(p_i)) = (0, r^n) \text{ and } (x_r(w_v^+), y_r(w_v^+)) = (-\infty, \infty),$$

we see that $x_r(\theta) < 0$ and $y_r(\theta) > 0$ for $\theta \in (w_v, p_i)$. On the other hand, by Lemma 3, (a, b) must lie on $S_v^{(r)}$, which is a contradiction since (a, b) is in the fourth quadrant of the plane while $S_v^{(r)}$ is in the second. Hence θ may belong to $(\xi_k, \xi_{k+1}) = (w_v, p_i) \subset (q_j, q_{j+1})$ only when i is odd, v, j are even and $i < n, j < m$.

(b) In case (C2), $\theta \in (\xi_k, \xi_{k+1}) = (p_i, q_j)$ for some $p_i \in P \backslash \{p_0\}$ and $q_j \in Q \backslash \{q_m\}$. By Lemma 2,

$$q_{j-1} < p_i < q_j < p_{i+1},$$

and

$$w_v < p_i < q_j < w_{v+1}$$

for some $w_v, w_{v+1} \in W$. Since $a > 0$, $b < 0$ and $\sin(n\theta)$, $\sin(m\theta)$ and $\sin(n - m)\theta$ have the same sign by (12), we see that if i is even, then j is odd and v is even. In this case, since

$$(x_r(p_i), y_r(p_i)) = (0, r^n) \text{ and } (x_r(q_j), y_r(q_j)) = (-r^m, 0),$$

we see that $x_r(\theta) < 0$ and $y_r(\theta) > 0$ for $\theta \in (p_i, q_j)$. On the other hand, by Lemma 3, (a, b) must lie on the curve $S_v^{(r)}$, which is a contradiction since (a, b) is in the fourth quadrant while $S_v^{(r)}$ is in the second. Hence θ may belong to the interval $(\xi_k, \xi_{k+1}) = (p_i, q_j) \subset I_v$ only when $i > 0$ is odd, $v < n - m$ is odd and j is even.

(c) In case (C3), we may similarly show that θ may belong to $(\xi_k, \xi_{k+1}) = (q_j, p_i) \subset I_v$ only when $i < n$ is odd, $j > 0$ is even and $v < n - m$ is even.

(d) In case (C4), we may similarly show that θ may belong to $(\xi_k, \xi_{k+1}) = (p_i, w_v) \subset (q_j, q_{j+1})$ only when i and j are odd, v is even and $i > 0, j < m$.

We summarize the above discussions as follows.

LEMMA 4. Assume (H1) holds. If $re^{i\theta}$, where $r \geq 0$ and $\theta \in [0, \pi]$, is a root of the polynomial $P(\lambda|a, b)$ with $a > 0$ and $b < 0$, then $r > 0$ and θ must satisfy either one of the following conditions:

(i) $\theta \in \{0, \pi\}$.

(ii) $\theta \in (\xi_k, \xi_{k+1}) = (w_v, p_i) \subset (q_j, q_{j+1})$ for some odd i, even j, even v, and $i < n$, $j < m$.

(iii) $\theta \in (\xi_k, \xi_{k+1}) = (p_i, q_j) \subset I_v$ for some odd i, even j, even v, and $j < m$.

(iv) $\theta \in (\xi_k, \xi_{k+1}) = (q_j, p_i) \subset I_v$ for some odd i, even j, even v, and $i < n$, $j > 0$.

(v) $\theta \in (\xi_k, \xi_{k+1}) = (p_i, w_v) \subset (q_j, q_{j+1})$ for some odd i, odd j, even v, and $i > 0$, $j < m$.

If the conditions $a > 0$ and $b < 0$ are changed to $a < 0$ and $b > 0$, then symmetric arguments will lead us to the following result which is needed in the last Section.

LEMMA 4'. Assume (H1) holds. If $re^{i\theta}$, where $r \geq 0$ and $\theta \in [0, \pi]$, is a root of the polynomial $P(\lambda|a, b)$ with $a < 0$ and $b > 0$, then $r > 0$ and θ must satisfy either one of the following conditions:

(i) $\theta \in \{0, \pi\}$.

(ii) $\theta \in (\xi_k, \xi_{k+1}) = (w_v, p_i) \subset (q_j, q_{j+1})$ for some even i, odd j, even v, and $i < n$, $j < m$.

(iii) $\theta \in (\xi_k, \xi_{k+1}) = (p_i, q_j) \subset I_v$ for some even i, odd j, odd v, and $i > 0$, $j < m$.

(iv) $\theta \in (\xi_k, \xi_{k+1}) = (q_j, p_i) \subset I_v$ for some even i, odd j, even v, and $i < n$.

(v) $\theta \in (\xi_k, \xi_{k+1}) = (p_i, w_v) \subset (q_j, q_{j+1})$ for some even i, odd j, even v, and $i > 0$, $j < m$.

The next result shows that the curves $S_0^{(1)}, S_1^{(1)}, ..., S_{n-m-1}^{(1)}$ defined by (14) may intersect with each other only at four specific places and they do not have self intersections.

LEMMA 5. Assume (H1) holds. Let $(x_1(\theta), y_1(\theta))$ be defined by (13), that is,

$$x_1(\theta) = \frac{\sin(n\theta)}{\sin(n - m)\theta}, \ y_1(\theta) = \frac{-\sin(m\theta)}{\sin(n - m)\theta}, \ \theta \in [0, \pi] \backslash W. \tag{15}$$

If $\alpha = x_1(\theta_1) = x_1(\theta_2)$ and $\beta = y_1(\theta_1) = y_1(\theta_2)$ where $\theta_1 > \theta_2$, $\theta_1 \in I_{v_1}$ and $\theta_2 \in I_{v_2}$ for some $v_1, v_2 \in \{0, 1, ..., n - m - 1\}$ (v_1 and v_2 may be the same). Then $(\alpha, \beta) \in \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$.

PROOF. Assume $\theta_1 \in I_{v_1} \cap P$, then $x_1(\theta_1) = 0$ and $|y_1(\theta_1)| = 1$. Since $x_1(\theta_2) = x_1(\theta_1) = 0$ and $|y_1(\theta_2)| = |y_1(\theta_1)| = 1$, then $\theta_2 \in I_{v_2} \cap P$. Similarly, assume $\theta_2 \in I_{v_2} \cap P$,

then $\theta_1 \in I_{v_1} \cap P$. So $(\alpha, \beta) \in \{(0,1), (0,-1), (1,0), (-1,0)\}$. Assume $\theta_1 \in I_{v_1} \cap Q$, then $y_1(\theta_1) = 0$ and $|x_1(\theta_1)| = 1$. Since $y_1(\theta_2) = y_1(\theta_1) = 0$ and $|x_1(\theta_2)| = |x_1(\theta_1)| = 1$, then $\theta_2 \in I_{v_2} \cap Q$. Similarly, assume $\theta_2 \in I_{v_2} \cap Q$, then $\theta_1 \in I_{v_2} \cap Q$. So $(\alpha, \beta) \in \{(0,1), (0,-1), (1,0), (-1,0)\}$.

Assume $\theta_1 \in I_{v_1} \backslash \{P \cup Q\}$ and $\theta_2 \in I_{v_2} \backslash \{P \cup Q\}$. Then

$$\sin(n\theta_1)\sin(n\theta_2)\sin(m\theta_1)\sin(m\theta_2) \neq 0.$$

Since

$$x_1(\theta) = \cos(m\theta) - (\cos(n-m)\theta)\, y_1(\theta)$$

and

$$y_1(\theta) = \cos(n\theta) - (\cos(n-m)\theta)\, x_1(\theta),$$

so our assumptions on θ_1 and θ_2 imply

$$\cos(m\theta_1) - \cos(m\theta_2) = \beta\left(\cos(n-m)\theta_1 - \cos(n-m)\theta_2\right) \tag{16}$$

and

$$\cos(n\theta_1) - \cos(n\theta_2) = \alpha\left(\cos(n-m)\theta_1 - \cos(n-m)\theta_2\right). \tag{17}$$

We need to consider two cases: (A) $\cos(n-m)\theta_1 \neq \cos(n-m)\theta_2$ and (B) $\cos(n-m)\theta_1 = \cos(n-m)\theta_2$.

(A) Consider first the case where $\cos(n-m)\theta_1 \neq \cos(n-m)\theta_2$. There are three subcases: (I) $\sin(m\theta_1) = \sin(m\theta_2)$; (II) $\sin(m\theta_1) = -\sin(m\theta_2) \neq 0$; (III) $\sin^2(m\theta_1) \neq \sin^2(m\theta_2)$.

(I) Assume $\sin(m\theta_1) = \sin(m\theta_2)$. Since $x_1(\theta_1) = x_1(\theta_2)$ and $y_1(\theta_1) = y_1(\theta_2)$, we see that $\sin(n-m)\theta_1 = \sin(n-m)\theta_2$ and $\sin(n\theta_1) = \sin(n\theta_2)$. Since

$$\begin{aligned}
\sin(n\theta_1)\cos(m\theta_1) - \cos(n\theta_1)\sin(m\theta_1) &= \sin(n-m)\theta_1 \\
&= \sin(n-m)\theta_2 \\
&= \sin(n\theta_2)\cos(m\theta_2) - \cos(n\theta_2)\sin(m\theta_2)
\end{aligned}$$

and

$$\sin(n\theta_1) = \sin(n\theta_2),$$

we see that

$$\sin(n\theta_1)[\cos(m\theta_1) - \cos(m\theta_2)] = \sin(m\theta_1)[\cos(n\theta_1) - \cos(n\theta_2)].$$

By (16) and (17), we have

$$\beta\sin(n\theta_1) = \alpha\sin(m\theta_1),$$

thus by substituting $\beta = -\sin(m\theta_1)/\sin(n-m)\theta_1$ and $\alpha = \sin(n\theta_1)/\sin(n-m)\theta_1$ into the above equation, we obtain $2\sin(n\theta_1)\sin(m\theta_1) = 0$. Thus $\theta_1 \in P$ or $\theta_1 \in Q$. It is a contradiction.

(II) Assume $\sin(m\theta_1) = -\sin(m\theta_2)$. Since $x_1(\theta_1) = x_1(\theta_2)$ and $y_1(\theta_1) = y_1(\theta_2)$, we have $\sin(n-m)\theta_1 = -\sin(n-m)\theta_2$ and $\sin(n\theta_1) = -\sin(n\theta_2)$. Since $\sin(n-m)\theta_1 = -\sin(n-m)\theta_2$ and $\sin(n\theta_1) = -\sin(n\theta_2)$, we have

$$\sin(n\theta_1)[\cos(m\theta_1) - \cos(m\theta_2)] = \sin(m\theta_1)[\cos(n\theta_1) - \cos(n\theta_2)].$$

Similar to Case 1, we may easily see that $\theta_1 \in P$ or $\theta_1 \in Q$. It is a contradiction.

(III) Assume $\sin^2(m\theta_1) \neq \sin^2(m\theta_2)$. By (16) and (17), we have

$$\left(\frac{\cos(m\theta_1) - \cos(m\theta_2)}{\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2)}\right)^2 = \beta^2 = \left(\frac{\sin(m\theta_1)}{\sin(n-m)\theta_1}\right)^2 = \left(\frac{\sin(m\theta_2)}{\sin(n-m)\theta_2}\right)^2 \tag{18}$$

and

$$\left(\frac{\cos(n\theta_1) - \cos(n\theta_2)}{\cos(n-m)\theta_1 - \cos(n-m)\theta_2}\right)^2 = \alpha^2 = \left(\frac{\sin(n\theta_1)}{\sin(n-m)\theta_1}\right)^2 = \left(\frac{\sin(n\theta_2)}{\sin(n-m)\theta_2}\right)^2. \tag{19}$$

Since $\sin^2(m\theta_1) \neq \sin^2(m\theta_2)$, by (18) and (19), we have $\sin^2(n\theta_1) \neq \sin^2(n\theta_2)$, $\sin^2(n-m)\theta_1 \neq \sin^2(n-m)\theta_1$,

$$\left(\frac{\cos(m\theta_1) - \cos(m\theta_2)}{\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2)}\right)^2$$

$$= \frac{\sin^2(m\theta_1) - \sin^2(m\theta_2)}{\sin^2(n-m)\theta_1 - \sin^2(n-m)\theta_2}$$

$$= \frac{\cos^2(m\theta_1) - \cos^2(m\theta_2)}{\cos^2(n-m)\theta_1 - \cos^2(n-m)\theta_2}$$

$$= \frac{\cos(m\theta_1) + \cos(m\theta_2)}{\cos(n-m)\theta_1 + \cos(n-m)\theta_2} \frac{\cos(m\theta_1) - \cos(m\theta_2)}{\cos(n-m)\theta_1 - \cos(n-m)\theta_2} \tag{20}$$

and

$$\left(\frac{\cos(n\theta_1) - \cos(n\theta_2)}{\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2)}\right)^2$$

$$= \frac{\sin^2(n\theta_1) - \sin^2(n\theta_2)}{\sin^2(n-m)\theta_1 - \sin^2(n-m)\theta_2}$$

$$= \frac{\cos^2(n\theta_1) - \cos^2(n\theta_2)}{\cos^2(n-m)\theta_1 - \cos^2(n-m)\theta_2}$$

$$= \frac{\cos(n\theta_1) + \cos(n\theta_2)}{\cos(n-m)\theta_1 + \cos(n-m)\theta_2} \frac{\cos(n\theta_1) - \cos(n\theta_2)}{\cos(n-m)\theta_1 - \cos(n-m)\theta_2} \tag{21}$$

If $\cos(m\theta_1) - \cos(m\theta_2) = 0$, then by (16), $\beta = 0$. So $\theta_1 \in Q$, which is a contradiction. If $\cos(n\theta_1) - \cos(n\theta_2) = 0$, by (17), $\alpha = 0$. So $\theta_1 \in P$, which is a contradiction. Thus we may assume $(\cos(m\theta_1) - \cos(m\theta_2))(\cos(n\theta_1) - \cos(n\theta_2)) \neq 0$. By (20) and (21), we see that

$$\frac{\cos(m\theta_1) - \cos(m\theta_2)}{\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2)} = \frac{\cos(m\theta_1) + \cos(m\theta_2)}{\cos(n\theta_1 - m\theta_1) + \cos(n\theta_2 - m\theta_2)}, \tag{22}$$

and

$$\frac{\cos(n\theta_1) - \cos(n\theta_2)}{\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2)} = \frac{\cos(n\theta_1) + \cos(n\theta_2)}{\cos(n\theta_1 - m\theta_1) + \cos(n\theta_2 - m\theta_2)}. \tag{23}$$

There are now three cases: (i) $\cos(m\theta_1) - \cos(m\theta_2) = \cos(m\theta_1) + \cos(m\theta_2)$; (ii) $\cos(n\theta_1) - \cos(n\theta_2) = \cos(n\theta_1) + \cos(n\theta_2)$; and (iii) $\cos(m\theta_1) - \cos(m\theta_2) \neq \cos(m\theta_1) + \cos(m\theta_2)$ and $\cos(n\theta_1) - \cos(n\theta_2) \neq \cos(n\theta_1) + \cos(n\theta_2)$. We assert that none of the three cases can be true:

(i) Assume $\cos(m\theta_1) - \cos(m\theta_2) = \cos(m\theta_1) + \cos(m\theta_2)$. Then $\cos(m\theta_2) = 0$. Furthermore, by (22), we have

$$\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2) = \cos(n\theta_1 - m\theta_1) + \cos(n\theta_2 - m\theta_2),$$

so that

$$\cos(n\theta_2 - m\theta_2) = 0.$$

But then $0 = \cos(n\theta_2 - m\theta_2) = \cos(n\theta_2)\cos(m\theta_2) + \sin(n\theta_2)\sin(m\theta_2) = \sin(n\theta_2)\sin(m\theta_2)$.. This is a contradiction.

(ii) Assume $\cos(n\theta_1) - \cos(n\theta_2) = \cos(n\theta_1) + \cos(n\theta_2)$. Then $\cos(n\theta_2) = 0$. Furthermore, by (23), we have

$$\cos(n\theta_1 - m\theta_1) - \cos(n\theta_2 - m\theta_2) = \cos(n\theta_1 - m\theta_1) + \cos(n\theta_2 - m\theta_2),$$

so that

$$\cos(n\theta_2 - m\theta_2) = 0.$$

But then $0 = \cos(n\theta_2 - m\theta_2) = \cos(n\theta_2)\cos(m\theta_2) + \sin(n\theta_2)\sin(m\theta_2) = \sin(n\theta_2)\sin(m\theta_2)$. This is a contradiction.

(iii) Assume $\cos(m\theta_1) - \cos(m\theta_2) \neq \cos(m\theta_1) + \cos(m\theta_2)$ and $\cos(n\theta_1) - \cos(n\theta_2) \neq \cos(n\theta_1) + \cos(n\theta_2)$. By (22) and (23), then

$$\frac{\cos(m\theta_2)}{\cos(n-m)\theta_2} = \beta = \frac{-\sin(m\theta_2)}{\sin(n-m)\theta_2},$$

so that

$$\sin(n-m)\theta_2 \cos(m\theta_2) + \sin(m\theta_2)\cos(n-m)\theta_2 = 0.$$

Then $\sin(n\theta_2) = 0$, which is a contradiction.

(B) Next we consider the case $\cos(n-m)\theta_1 = \cos(n-m)\theta_2$. By (16) and (17), $\cos(m\theta_1) = \cos(m\theta_2)$ and $\cos(n\theta_1) = \cos(n\theta_2)$. Thus,

$$\sin^2(m\theta_1) = 1 - \cos^2(m\theta_1) = 1 - \cos^2(m\theta_2) = \sin^2(m\theta_2).$$

There are two cases: (a) $\sin(m\theta_1) = \sin(m\theta_2)$; and (b) $\sin(m\theta_1) = -\sin(m\theta_2)$. Neither cases can be true:

(a) Assume $\sin(m\theta_1) = \sin(m\theta_2)$. Since $x_1(\theta_1) = x_1(\theta_2)$ and $y_1(\theta_1) = y_1(\theta_2)$, we have $\sin(n\theta_1) = \sin(n\theta_2)$. Since $\cos(m\theta_1) = \cos(m\theta_2)$, $\sin(m\theta_1) = \sin(m\theta_2)$, $\cos(n\theta_1) = \cos(n\theta_2)$, and $\sin(n\theta_1) = \sin(n\theta_2)$, we see that

$$\sin(m\theta_1 - m\theta_2) = \sin(m\theta_1)\cos(m\theta_2) - \cos(m\theta_1)\sin(m\theta_2) = 0$$

and

$$\sin(n\theta_1 - n\theta_2) = \sin(n\theta_1)\cos(n\theta_2) - \cos(n\theta_1)\sin(n\theta_2) = 0.$$

So $\theta_1 - \theta_2 = q_j = p_i$ for some $j = 1, ..., m-1$ and $j = 1, ..., n-1$. It is a contradiction since $q_j \neq p_i$ for any $j = 1, ..., m-1$ and $j = 1, ..., n-1$.

(b) Assume $\sin(m\theta_1) = -\sin(m\theta_1)$. Since $x_1(\theta_1) = x_1(\theta_2)$ and $y_1(\theta_1) = y_1(\theta_2)$, we see that $\sin(n\theta_1) = -\sin(n\theta_2)$. Since $\cos(m\theta_1) = \cos(m\theta_2)$, $\sin(m\theta_1) = -\sin(m\theta_2)$, $\cos(n\theta_1) = \cos(n\theta_2)$, and $\sin(n\theta_1) = -\sin(n\theta_2)$, we see that

$$\sin(m\theta_1 + m\theta_2) = \sin(m\theta_1)\cos(m\theta_2) + \cos(m\theta_1)\sin(m\theta_2) = 0.$$

and

$$\sin(n\theta_1 + n\theta_2) = \sin(n\theta_1)\cos(n\theta_2) + \cos(n\theta_1)\sin(n\theta_2) = 0.$$

Thus $\theta_1 + \theta_2 = q_j = p_i$ for some $j = 1, ..., m$ and $i = 1, ..., n$. Since $q_j \neq p_i$ for any $j = 1, ..., m-1$ and $j = 1, ..., n-1$, then $\theta_1 + \theta_2 = \pi$. Since $\sin(m\theta_2) = \sin(-m\theta_1) = \sin(m\theta_2 - m\pi) \neq 0$ and $\sin(n\theta_2) = \sin(-n\theta_1) = \sin(n\theta_2 - n\pi) \neq 0$, then n and m are even. It is a contradiction since $\gcd(n, m) = 1$. The proof is complete.

The next result shows that $S_0^{(1)}$, $S_1^{(1)}$, ..., $S_{n-m-2}^{(1)}$ and $L_1^{(1)}$, $L_2^{(1)}$ can intersect at specific places only.

LEMMA 6. Assume (H1) holds. Let $L_1^{(1)}$ and $L_2^{(1)}$ be the straight lines defined by (10) and (11) respectively, that is,

$$L_1^{(1)} = \{(x, y) \in R^2 : 1 - x - y = 0\},$$

and

$$L_2^{(1)} = \{(x, y) \in R^2 : \cos(n\pi) - x\cos(n-m)\pi - y = 0\},$$

and let the curves $S_0^{(r)}$, $S_1^{(r)}$, ..., $S_{n-m-2}^{(r)}$ be defined by (14). Then $S_v^{(1)} \cap L_1^{(1)} \subseteq \{(0,1), (1,0)\}$ and (I) $S_v^{(1)} \cap L_2^{(1)} \subseteq \{(0,1), (-1,0)\}$ for any $v \in \{0, 1, ..., n-m-1\}$ under (H2), (II) $S_v^{(1)} \cap L_2^{(1)} \subseteq \{(-1,0), (0,-1)\}$ for any $v \in \{0, 1, ..., n-m-1\}$ under (H3), and (III) $S_v^{(1)} \cap L_2^{(1)} \subseteq \{(1,0), (0,-1)\}$ for any $v \in \{0, 1, ..., n-m-1\}$ under (H4)..

PROOF. We first show that $S_v^{(1)} \cap L_1^{(1)} \subseteq \{(0,1), (1,0)\}$ for $v \in \{0, 1, ..., n-m-1\}$. If there is $\theta \in [0, \pi] \setminus W$ such that

$$1 = x_1(\theta) + y_1(\theta) = \frac{\sin n\theta}{\sin(n-m)\theta} - \frac{\sin m\theta}{\sin(n-m)\theta}.$$

Then

$$\sin(n\theta) - \sin(m\theta) = \sin(n-m)\theta = \sin(n\theta)\cos(m\theta) - \cos(n\theta)\sin(m\theta), \qquad (24)$$

so that

$$\sin(n\theta)(1 - \cos(m\theta)) = \sin(m\theta)(1 - \cos(n\theta)). \qquad (25)$$

In case $\theta \notin P \cup Q$, then $1 - \cos m\theta \neq 0$, $1 - \cos n\theta \neq 0$, and

$$\left(\frac{1 - \cos(n\theta)}{1 - \cos(m\theta)}\right)^2 = \left(\frac{\sin(n\theta)}{\sin(m\theta)}\right)^2 = \left(\frac{1 - \cos(n\theta)}{1 - \cos(m\theta)}\right)\left(\frac{1 + \cos(n\theta)}{1 + \cos(m\theta)}\right)$$

so that

$$\frac{1 - \cos(n\theta)}{1 - \cos(m\theta)} = \frac{1 + \cos(n\theta)}{1 + \cos(m\theta)}. \tag{26}$$

We consider three cases: (i) $1 - \cos(n\theta) = 1 + \cos(n\theta)$; (ii) $1 - \cos(m\theta) = 1 + \cos(m\theta)$; and (iii) $1 - \cos(n\theta) \neq 1 + \cos(n\theta)$.

(i) Assume $1 - \cos(n\theta) = 1 + \cos(n\theta)$. Then $1 + \cos(m\theta) = 1 - \cos(m\theta)$ by (26) and $\cos(n\theta) = \cos(m\theta) = 0$. By (24), we see that $\sin(n - m)\theta = 0$, which is contrary to $\theta \in [0, \pi] \backslash W$.

(ii) is similar to the previous case.

(iii) Assume $1 - \cos(n\theta) \neq 1 + \cos(n\theta)$, then $1 - \cos(m\theta) \neq 1 + \cos(m\theta)$ by (26). Since $1 - \cos(n\theta) \neq -(1 + \cos(n\theta))$ and $1 - \cos(m\theta) \neq -(1 + \cos(m\theta))$, thus

$$1 = \frac{\cos(n\theta)}{\cos(m\theta)}$$

by (26). By (25), we have $\sin(n\theta) = \sin(m\theta)$. So $\sin(n - m)\theta = 0$ by (24), which is contrary to $\theta \in [0, \pi] \backslash W$.

We may now see that $\theta \in P \cup Q \backslash W$. If $\theta = p_i$ for some $i \in \{1, 2, ..., n - 1\}$, since $x_1(\theta) + y_1(\theta) = 1$, then i is even. Thus $x_1(\theta) = 0$ and $y_1(\theta) = 1$. If $\theta = q_j$ for some $j \in \{1, 2, ..., m - 1\}$, then since $x_1(\theta) + y_1(\theta) = 1$, we see that j is odd. Thus $x_1(\theta) = 1$ and $y_1(\theta) = 0$. So $S_v^{(1)} \cap L_1^{(1)} \subseteq \{(0, 1), (-1, 0)\}$ for any $v \in \{0, 1, ..., n - m - 1\}$.

Now, we prove $S_v^{(1)} \cap L_2^{(1)} \subseteq \{(0, -1), (1, 0), (0, 1), (-1, 0)\}$ for any $v \in \{0, 1, ..., n - m - 1\}$. It turns out that the parities of n and m matter. For this reason, we consider three cases: (a) (H2) holds; (b) (H3) holds; and (c) (H4) holds.

(a) Suppose n is even and m is odd. Then

$$L_2^{(1)} = \{(x, y) \in R^2 : 1 + x - y = 0\}.$$

If there is $\theta \in [0, \pi] \backslash W$ such that $x_1(\theta) - y_1(\theta) = -1$, then

$$\sin(n\theta) + \sin(m\theta) = -\sin(n - m)\theta = -\sin(n\theta)\cos(m\theta) + \cos(n\theta)\sin(m\theta), \tag{27}$$

so that

$$\sin(n\theta)(1 + \cos(m\theta)) = \sin(m\theta)(\cos(n\theta) - 1). \tag{28}$$

If $\theta \notin P \cup Q$, then

$$\left(\frac{1 - \cos(n\theta)}{1 + \cos(m\theta)}\right)^2 = \left(\frac{\sin(n\theta)}{\sin(m\theta)}\right)^2 = \left(\frac{1 + \cos(n\theta)}{1 - \cos(m\theta)}\right)\left(\frac{1 - \cos(n\theta)}{1 + \cos(m\theta)}\right)$$

and

$$\frac{1 - \cos(n\theta)}{1 + \cos(m\theta)} = \frac{1 + \cos(n\theta)}{1 - \cos(m\theta)}. \tag{29}$$

We consider three cases: (i) $1 - \cos(n\theta) = 1 + \cos(n\theta)$; (ii) $1 - \cos(m\theta) = 1 + \cos(m\theta)$; and (iii) $1 - \cos(n\theta) \neq 1 + \cos(n\theta)$.

(i) Assume $1 - \cos(n\theta) = 1 + \cos(n\theta)$. Then $1 + \cos(m\theta) = 1 - \cos(m\theta)$ by (29) and $\cos(n\theta) = \cos(m\theta) = 0$. By (27), $\sin(n - m)\theta = 0$, which is a contradiction.

(ii) is similar to the previous case.

(iii) Assume $1 - \cos(n\theta) \neq 1 + \cos(n\theta)$. Then $1 - \cos(m\theta) \neq 1 + \cos(m\theta)$ by (29). Since $1 - \cos(n\theta) \neq -(1 + \cos(n\theta))$ and $1 + \cos(m\theta) \neq -(1 - \cos m\theta)$, then

$$1 = -\frac{\cos(n\theta)}{\cos(m\theta)}$$

by (29). Then $\cos(n\theta) = -\cos(m\theta)$. By (28), $\sin(n\theta) = -\sin(m\theta)$. So $\sin(n-m)\theta = 0$ by (27), which is a contradiction.

By the previous discussions, we see that $\theta \in (P \cup Q)\backslash W$. If $\theta = p_i$ for some $i \in \{1, 2, ..., n-1\}$, then since $x_1(\theta) - y_1(\theta) = -1$, we see that i is even. Thus $x_1(\theta) = 0$ and $y_1(\theta) = 1$. If $\theta = q_j$ for some $j \in \{1, 2, ..., m-1\}$, then since $x_1(\theta) - y_1(\theta) = -1$, we see that j is odd. Thus $x_1(\theta) = 1$ and $y_1(\theta) = 0$. So $S_v^{(1)} \cap L_2^{(1)} \subseteq \{(0,1), (-1,0)\}$, for any $v \in \{0, 1, ..., n-m-1\}$.

The proof of the other two cases (H3) and (H4) are similar and hence omitted. The proof is complete.

In view of Figure 1, we may observe that the parametric function $(x_1(t), y_1(t))$, when used to describe a moving particle, will trace out trajectories in definite patterns. Such behaviors will be useful for our future discussions. We will base our description of the corresponding behaviors on Lemmas 4 and 4'.

LEMMA 7. Assume (H1) holds. Let $F(t) = (x_1(t), y_1(t))$ be the parametric function defined by (15). Let $I = (\xi_k, \xi_{k+1})$ be one of the constituent open intervals of $[0, \pi]\backslash W$.

(i) If $I = (p_0, p_1) = (0, \pi/n)$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the fourth quadrant with initial point $(\frac{n}{n-m}, \frac{-m}{n-m})$ and endpoint $(0, -1)$.

(ii) If $I = (w_v, p_i)$, where i is odd, v is even, $i < n$ and $v > 0$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the fourth quadrant with 'initial point $(\infty, -\infty)$' and endpoint $(0, -1)$.

(iii) If $I = (p_i, w_v)$, where i is odd, v is even, $i > 0$ and $v < n - m$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the fourth quadrant with 'initial point $(0, -1)$ and 'endpoint $(\infty, -\infty)$'.

(iv) If $I = (p_i, q_j)$, where i is odd, j is even, $i > 0$ and $j < m$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the fourth quadrant with initial point $(0, -1)$ and endpoint $(1, 0)$.

(v) If $I = (q_j, p_i)$, where i is odd, j is even, $i < n$ and $j > 0$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the fourth quadrant with initial point $(1, 0)$ and endpoint $(0, -1)$.

(vi) If $I = (p_{n-1}, \pi)$, then (a) the restriction $F|_I$ traces out a (directed) curve in the interior of the third quadrant with initial point $(0, -1)$ and endpoint $(\frac{-n}{n-m}, \frac{-m}{n-m})$ under (H2); (b) then the restriction $F|_I$ traces out a (directed) curve in the interior of the second quadrant with initial point $(0, 1)$ and endpoint $(\frac{-n}{n-m}, \frac{m}{n-m})$ under (H3).

PROOF. First note that $x_1(t)y_1(t) \neq 0$ for $t \in I$. Thus the directed path traced out by each $F|_{I_v}$ lies in one of the first, second, third or fourth quadrant of the plane. Since

$$\lim_{t \to 0^+} x_1(t) = \frac{n}{n-m} > 0, \ \lim_{t \to 0^+} y_1(t) = -\frac{m}{n-m} < 0, \tag{30}$$

and

$$(x_1(p_1), y_1(p_1)) = \left(x_1\left(\frac{\pi}{n}\right), y_1\left(\frac{\pi}{n}\right) \right) = (0, -1), \tag{31}$$

we see that $x_1(t) > 0$ and $y_1(t) < 0$ for $t \in (p_0, p_1)$. That is, $F|_{(p_0, p_1)}$ traces out a path in the fourth quadrant with initial point $(\frac{n}{n-m}, \frac{-m}{n-m})$ and endpoint $(0, -1)$.

The case (vi) is similarly proved.

Since $(x(p_i), y(p_i)) = (0, -1)$ and $(x(q_j), y(q_j)) = (1, 0)$ for every odd i and even j, the proofs of (i), (iv) and (v) follow from the same reason shown above.

If $I = (w_v, p_i)$ for some odd i and even v, then

$$\lim_{\theta \to w_v^+} x_1(\theta) = \infty \text{ and } \lim_{\theta \to w_v^+} y_1(\theta) = -\infty.$$

If $I = (p_i, w_v)$ for some odd i and even v, then

$$\lim_{\theta \to w_v^-} x_1(\theta) = \infty \text{ and } \lim_{\theta \to w_v^-} y_1(\theta) = -\infty.$$

By reasoning similar to that used in the first case, the proofs of (ii) and (iii) follows. The proof is complete.

4   Additional Properties of Normal Root Curves under (H2)

Under the assumption that n is even and m is odd, we will be concerned with several additional properties of the curves $L_1^{(r)}$, $L_2^{(r)}$, $S_0^{(r)}$, $S_1^{(r)}$, ..., $S_{n-m-2}^{(r)}$ or $S_{n-m-1}^{(r)}$ when $r = 1$. First, note that under the assumption (H2), $L_2^{(1)}$ is now of the form

$$L_2^{(1)} = \{(x, y) \in R^2 : 1 + x - y = 0\}.$$

Assume (H1) holds. Since (30) and (31) hold, and since

$$\lim_{t \to 0^+} x_1(t) + \lim_{t \to 0^+} y_1(t) = \frac{n}{n-m} - \frac{m}{n-m} = 1,$$

we see that the parametric function $(x_1(t), y_1(t))$ when restricted to the interval $(0, \pi/n)$ traces out a (directed) curve $S_0^{(1)}|_{(0,\pi/n)}$ in the fourth quadrant with initial point $(\frac{n}{n-m}, -\frac{m}{n-m}) \in L_1^{(1)}$ and end point $(0, -1)$. We let Ω_0 be the set of points strictly inside the fourth quadrant bounded by the x-axis, y-axis, L_1 and the curve $S_0^{(1)}|_{(0,\pi/n)}$. For example, the case where $n = 8$ and $m = 1$ is depicted in Figure 2.

LEMMA 8. Assume (H1) and (H2) hold and $m = 1$. Let Ω_0 be the set of points in the interior of the fourth quadrant bounded by the x-axis, y-axis, $L_1^{(1)}$ and $S_0^{(1)}|_{(0,\pi/n)}$. Then Ω_0 does not intersect with the curves $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}$, ..., $S_{n-m-2}^{(1)}$ and $S_{n-m-1}^{(1)}$.

PROOF. The definitions of Ω_0 and $L_2^{(1)}$ show that no part of $L_2^{(1)}$ lies inside Ω_0. Since $L_1^{(1)}$ and $S_0^{(1)}$ does not intersect by Lemma 6, no points of $L_1^{(1)}$ can lie inside Ω_0
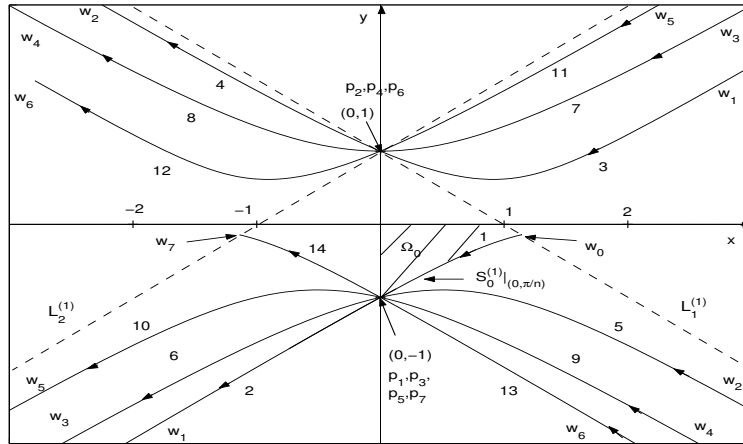
Figure 2: $n = 8, m = 1$

neither. If there is a point $(a, b) \in S_{v'}^{(1)}$, where $v' \in \{0, 1, ..., n-m-1\}$ and $(a, b) \in \Omega_0$, then $a = x_1(t')$ and $b = y_1(t')$ for some $t' \in I_{v'}$. Under the condition $m = 1$, we must have $Q = \{0, \pi\}$. By Lemma 4, t' must satisfy either (1) $t' \in (\xi_k, \xi_{k+1}) = (w_v, p_i)$ for some odd $i < n$ and some even v, or (2) $t' \in (\xi_k, \xi_{k+1}) = (p_i, w_v)$ for some odd $i > 0$ and even $v < n - m$, or (3) $t' \in (\xi_k, \xi_{k+1}) = (p_{n-1}, \pi)$.

If t' satisfies the first condition, by Lemma 7, the curve $S_{v'}^{(1)}$ over the interval $(\xi_k, \xi_{k+1}) = (w_v, p_i)$ and the curve $S_0^{(1)}$ over the interval $(0, p_1)$ intersect with each other at (a, b) which is in the interior of the fourth quadrant. This is a contradiction by Lemma 5. Similarly, the other two conditions will also lead to a contradiction. The proof is complete.

Assume (H1) and (H2) hold. Suppose $m \geq 3$ (see for example Figure 1). Then for each q_j where $j \in \{2, 4, ..., m-1\}$, by Lemma 2, there is $v \in \{0, 1, ..., n-m-1\}$ such that $q_j \in I_v$. We consider the slopes of the graphs of the parametric function $(x_1(t), y_1(t))$ at $t = q_j$. First note that

$$\frac{dy_1}{dx_1}(t) = \frac{n \sin(mt) \cos(nt - mt) - m \sin(nt)}{m \sin(nt) \cos(nt - mt) + n \sin(mt)}.$$

Since $\gcd(n, m) = 1$ and m is odd, we see that $\sin(nq_j) \cos(nq_j) \neq 0$ for $j \in \{2, 4, ..., m-1\}$ and

$$\frac{dy_1}{dx_1}(q_j) = -\frac{1}{\cos(nq_j)} \text{ for } j = 2, 4, ..., m-1.$$

We assert that $\cos(nq_2), ..., \cos(nq_{m-1})$ are mutually distinct. Indeed, if there are distinct j_1 and j_2 in $\{2, 4, ..., m-1\}$ such that $\cos(nq_{j_1}) = \cos(nq_{j_2})$, then $nj_1 = \pm nj_2 \mod m$. Since $\gcd(n, m) = 1$, then $j_1 = \pm j_2 \mod m$. If $j_1 = j_2 \mod m$, then since $0 < j_1, j_2 < m$, we have $j_1 = j_2$, which is a contradiction. If $j_1 = -j_2 \mod m$, then since $0 < j_1, j_2 < m$, we have $j_1 = m - j_2$, which is contrary to the fact that $j_1 + j_2$ is even and m is odd.

Among the derivatives $\frac{dy_1}{dx_1}(q_2), \frac{dy_1}{dx_1}(q_4), ..., \frac{dy_1}{dx_1}(q_{m-1})$, let us collect those which are positive and place them in a set Γ. Next, we let $q_{j^*} \in Q$ where $j^* \in \{2, 4, ..., m-1\}$ such that

$$\frac{dy_1}{dx_1}(q_{j^*}) = \begin{cases} \min \Gamma & \text{if } \Gamma \neq \emptyset \\ \min \left\{ \frac{dy_1}{dx_1}(q_j) : j = 2, 4, ..., m-1 \right\} & \text{if } \Gamma = \emptyset \end{cases} . \tag{32}$$

Since $q_{j^*} = \xi_k$ for some $\xi_k \in P \cup Q \cup W$ and $j^* \in \{2, 4, ..., m-1\}$, by Lemma 2, there are p_{i^*} and p_{i^*+1} in P which are closest to q_{j^*} and $p_{i^*} = \xi_{k-1} < \xi_k < \xi_{k+1} = p_{i^*+1}$. If i^* is odd, by Lemma 7(iv), the parametric function $(x_1(t), y_1(t))$ over the interval $(\xi_{k-1}, \xi_k) = (p_{i^*}, q_{j^*})$ traces out a (directed) curve in the interior of the fourth quadrant with initial point $(0, -1)$ and endpoint $(1, 0)$. Similarly, if i^* is even, then by Lemma 7(v), the parametric function $(x_1(t), y_1(t))$ over the interval $(\xi_k, \xi_{k+1}) = (q_{j^*}, p_{i^*+1})$ traces out a (directed) curve in the interior of the fourth quadrant of the plane with initial point $(1, 0)$ and endpoint $(0, -1)$. We will now let J be either (ξ_{k-1}, ξ_k) or (ξ_k, ξ_{k+1}) defined above. We will also let Ω_* be the set of points strictly inside the fourth quadrant and bounded by the x-axis, y-axis and the curve traced out by $(x_1(t), y_1(t))$ over the interval J.

LEMMA 9. Assume (H1) and (H2) hold and $m \geq 3$. Let

$$J = \begin{cases} (p_{i^*}, q_{j^*}) & \text{if } i^* \text{ is odd} \\ (q_{j^*}, p_{i^*+1}) & \text{if } i^* \text{ is even} \end{cases} ,$$

where $j^* \in \{2, 4, ..., m-1\}$ is determined by (32) and p_{i^*}, p_{i^*+1} are points in P that are closest to q_{j^*}, and let S be the curve traced out by the parametric function $(x_1(t), y_1(t))$ over the interval J. Then the set of points Ω_* in the interior of the fourth quadrant bounded by the x-axis, y-axis and the curve S cannot contain any points of $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}, ..., S_{n-m-2}^{(1)}$ and $S_{n-m-1}^{(1)}$.

PROOF. The definition of Ω_* excludes any part of $L_2^{(1)}$. Since $L_1^{(1)}$ and S does not intersect, no points of $L_1^{(1)}$ can lie inside Ω_* neither. Let $(a, b) \in S_{v'}^{(1)}$ where $v' \in \{0, 1, ..., n-m-1\}$ and $(a, b) \in \Omega_*$. Then $a = x_1(t')$ and $b = y_1(t')$ for some $t' \in I_{v'}$. Since $a > 0$ and $b < 0$, by Lemma 4, t' must satisfy either one of the following conditions: (1) $t' \in (\xi_k, \xi_{k+1}) = (w_v, p_i)$ for some odd $i < n$ and even v, (2) $t' \in (\xi_k, \xi_{k+1}) = (p_i, w_v)$ for some odd i and even $v < n-m$, (3) $t' \in (\xi_k, \xi_{k+1}) = (p_i, q_j)$ for some odd $i > 0$ and even $j < m$, or, (4) $t' \in (\xi_k, \xi_{k+1}) = (q_j, p_i)$ for some odd $i < n$ and even $j > 0$..

If t' satisfies condition (1), then by Lemma 7, the path traced out by $(x_1(t), y_1(t))$ over the interval $(\xi_k, \xi_{k+1}) = (w_v, p_i)$ and the path over the interval J intersects with each other at (a, b) which is in the interior of the fourth quadrant. This is a contradiction by Lemma 5. Similarly, the condition (2) leads to a contradiction.

If t' satisfies the condition (3), then by Lemma 5 and Lemma 7, the parametric function $(x_1(t), y_1(t))$ over the interval $(\xi_k, \xi_{k+1}) = (p_i, q_j)$ traces out a (directed) path in the Ω_* with initial point $(0, -1)$ and endpoint $(1, 0)$. If $\frac{dy_1}{dx_1}(q_{j^*}) > 0$, then

$$\frac{dy_1}{dx_1}(q_{j^*}) > \frac{dy_1}{dx_1}(q_j) > 0.$$

This is contradictory to the definition of q_{j*}. If $\frac{dy_1}{dx_1}(q_{j*}) < 0$, then

$$\frac{dy_1}{dx_1}(q_{j*}) < \frac{dy_1}{dx_1}(q_j) < 0.$$

This is contrary to the definition of q_{j*} again. Similarly, the condition (4) will lead to a contradiction. The proof is complete.

5   Exact Stability Region under (H2)

By Lemma 3, if $e^{i\theta}$, where $\theta \in [0, 2\pi]$ is a (normal) root of $P(\lambda|a, b) = 0$, then (a, b) must lie on the curves $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}$, ..., $S_{n-m-2}^{(1)}$ or $S_{n-m-1}^{(1)}$. The converse is also true.

LEMMA 10.   Suppose (H1) and (H2) hold. Then the polynomial $P(\lambda|a, b) = \lambda^n - a\lambda^{n-m} - b$, where $a, b \in R$, has a normal root if, and only if, (a, b) lies on the curves $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}$, ..., $S_{n-m-2}^{(1)}$ or $S_{n-m-1}^{(1)}$.

Indeed, if (a, b) lies on the curve $S_v^{(1)}$ for some $v \in \{0, 1, ..., n - m - 1\}$, then there is a $t^* \in I_v$ such that

$$a = \frac{\sin(nt^*)}{\sin(n-m)t^*}, \; b = \frac{-\sin(mt^*)}{\sin(n-m)t^*}.$$

Since $\sin(nt^* - mt^*) \neq 0$, it is then easily checked that $P(\lambda|a, b)$ has the normal root e^{it^*}. If (a, b) is a point in $L_1^{(1)}$, then

$$P(1|a, b) = 1 - a - b = 0.$$

Hence $P(\lambda|a, b)$ has the normal root 1. Similarly, if (a, b) lies on $L_2^{(1)}$, then the polynomial $P(\lambda|a, b)$ has the normal root -1.

Next, we observe that under (H1) and (H2), the stability region $\Omega(n, m)$ is symmetric with respect to the y-axis in the x, y-plane, that is,

$$(a, b) \in \Omega(n, m) \Leftrightarrow (-a, b) \in \Omega(n, m).$$

Indeed, this follows from the fact that when n is even and m is odd,

$$P(-\lambda|a, b) = (-\lambda)^n - a(-\lambda)^{n-m} - b = \lambda^n + a\lambda^{n-m} - b = P(\lambda| - a, b).$$

Therefore, we only need to characterize $\Omega(n, m)$ in the right half plane. For this purpose, we break the right half plane into five mutually exclusive and exhaustive subregions:

$$A = \{(x, y) \in R^2 : y \geq 1 - x, \; x \geq 0\}, \tag{33}$$

$$B = \{(0, y) \in R^2 : y < 1\}, \tag{34}$$

$$C = \{(x, 0) \in R^2 : 0 < x < 1\}, \tag{35}$$

$$D = \{(x, y) \in R^2 : y < 1 - x, \ x > 0, \ y > 0\}, \tag{36}$$

and

$$E = \{(x, y) \in R^2 : y < 1 - x, \ x > 0, \ y < 0\} \tag{37}$$

We assert that A is in the complement of $\Omega(n, m)$. Indeed, if $(a, b) \in A$, then $a + b \geq 1$ so that $P(1|a, b) = 1 - (a + b) \leq 0$. But since $\lim_{\lambda \to +\infty, \lambda \in R} P(\lambda) = +\infty$, we see that $P(\lambda|a, b)$ has a real root with modulus greater than 1.

Let B' be the subset of B defined by

$$B' = \{(0, y) \in R^2 : -1 < y < 1\}. \tag{38}$$

We assert that B' is part of $\Omega(n, m)$. Indeed, if $(a, b) \in B$, then $a = 0$ and $b \in R$ so that $P(\lambda|a, b) = \lambda^n - b$. Clearly, every root of P is subnormal if $|b| < 1$, and every root of P is normal of supernormal if $|b| \geq 1$, that is, $(0, b) \in B \backslash B'$.

We assert that C is part of $\Omega(n, m)$. Indeed, if $(a, b) \in C$, then $P(\lambda|a, b) = \lambda^{n-m}(\lambda^m - a)$. Again all roots of P are subnormal if $a \in (0, 1)$.

Next, we assert that D is part of $\Omega(n, m)$. Indeed, suppose $(a, b) \in D$, then $a > 0, b > 0$ and $a + b < 1$. If λ is a root of $P(\lambda|a, b) = \lambda^n - a\lambda^{n-m} - b$, then in view of $b > 0$, we have $\lambda \neq 0$. Hence $1 - a\lambda^{-m} - b\lambda^{-n} = 0$ which implies

$$a + b < 1 \leq a \left|\lambda^{-m}\right| + b \left|\lambda^{-n}\right|.$$

But then $a < a |\lambda|^{-m}$ or $b < b |\lambda|^{-n}$. In either cases, $|\lambda| < 1$ as required.

We are now ready for one of our main results.

THEOREM 1. Suppose (H1) and (H2) hold. If $m = 1$, let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval $(0, \pi/n)$. If $m \geq 3$, let

$$J = \begin{cases} (p_{i^*}, q_{j^*}) & \text{if } i^* \text{ is odd} \\ (q_{j^*}, p_{i^*+1}) & \text{if } i^* \text{ is even} \end{cases},$$

where $j^* \in \{2, 4, ..., m - 1\}$ is determined by (32) and p_{i^*}, p_{i^*+1} are points in P that are closest to q_{j^*}, and let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval J. Let $\widetilde{S}$ be the curve which is the symmetric image of S with respect to the y-axis. Then $\Omega(n, m)$ is the set of points in the interior of the (bounded) region bounded by $L_1^{(1)}$, $L_2^{(1)}$ and the curve $S \cup \{(0, 0)\} \cup \widetilde{S}$.

PROOF. We will assume that $m = 1$, since the case where $m \geq 3$ is similar. We have already shown in Lemma 7 that the curve S separates the region E into three parts Ω_0, S and E' where Ω_0 is the set of points in E bounded by the x-axis, y-axis, $L_1^{(1)}$ and S; and E' is the complement $E \backslash (\Omega_0 \cup S)$. We assert that Ω_0 is part of $\Omega(n, 1)$. Indeed, if $(a, b) \in \Omega_0$ but $\rho(a, b) \geq 1$, then since $\rho(0, 0) = 0$ and since we can joint the points $(0, 0)$ and (a, b) by means of a continuous curve contained completely in Ω_0, by continuity, there is some point $(\alpha, \beta) \in \Omega_0$ such that $\rho(\alpha, \beta) = 1$. But then (α, β) must lie in one of the curves $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}$, ..., $S_{n-m-2}^{(1)}$ or $S_{n-m-1}^{(1)}$. This is contrary to the conclusion of Lemma 8.

Next, each point $(\mu, \nu) \in S$ satisfies $\rho(\mu, \nu) \geq 1$, so that S is outside $\Omega(n, 1)$.

Next, we assert that E' is outside $\Omega(n,1)$. Indeed, let $(a,b) \in E'$. Consider the parametric curve

$$x(r) = \frac{a}{r^m}, \; y(r) = \frac{b}{r^n}, \; r \in [1,\infty).$$

Clearly, it traces out a (directed) curve inside the interior of the fourth open quadrant with 'initial point' (a,b) and 'endpoint' $(0,0)$. Hence it intersects either with S or with $L_1^{(1)}$. Therefore, there exists a $\bar{r} > 1$ such that $(a/\bar{r}^m, b/\bar{r}^n) \in S$ or $(a/\bar{r}^m, b/\bar{r}^n) \in L_1^{(1)}$. In the former case, $(a/\bar{r}^m, b/\bar{r}^n) = (x_1(\theta), y_1(\theta))$ for some $\theta \in J$, so that $a = x_1(\theta)\bar{r}^m$ and $b = y_1(\theta)\bar{r}^n$. Since $x_1(\theta)r^m = x_r(\theta)$ and $y_1(\theta)r^n = y_r(\theta)$ (see (12)), thus it is easily seen that $P(\bar{r}e^{i\theta}|a,b) = 0$. That is, $P(\lambda|a,b)$ has a root with absolute value strictly greater than 1. In the latter case, $a/\bar{r}^m + b/\bar{r}^n = 1$, so that $P(\bar{r}|a,b) = 0$. Again, P has a real root with absolute value greater than 1. Our assertion is thus proved.

According to the previous discussions, we see that in the closed right half plane, every point in the union $B' \cup D \cup C \cup \Omega_0$ is stable while every point in the rest of the closed right half plane is not. Finally, our proof follows from the symmetry of $\Omega(n,1)$ with respect to the y-axis. The proof is complete.

The case where $n = 8$ and $m = 1$ is illustrated in Figure 3.
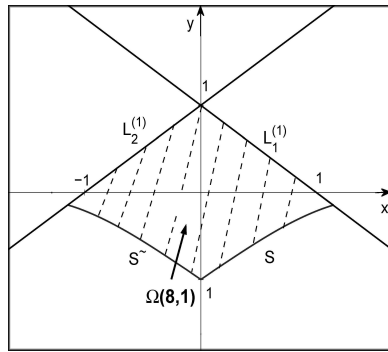

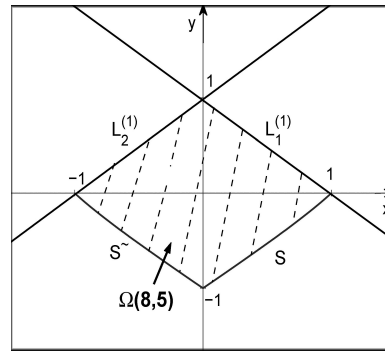
Figure 3: $n = 8, m = 1$ Figure 4: $n = 8, m = 5$

To illustrate Theorem 1 for the case $m \geq 3$, let us consider the case where $n = 8$ and $m = 5$. Since $q_j = j\pi/5$ for $j = 0, 1, 2, 3, 4, 5$, we see that

$$-\frac{1}{\cos(8q_2)} = -3.2361...$$
$$-\frac{1}{\cos(8q_4)} = 1.2361...$$

so that $j^* = 2$. In view of (5), we see that p_3 and p_4 are closest to q_2 and $p_3 < q_2 < p_4$. Since 3 is odd, we see that $J = (p_3, q_2) = (3\pi/8, 2\pi/5)$. Thus

$$S = \{(x_1(\theta), y_1(\theta)) : \theta \in J\}$$

which together with $\tilde{S}$ and the stability region $\Omega(8,5)$ are depicted in Figure 4.

6 Exact Stability Region under (H3)

Under the assumption that n and m are odd, the preparatory results that lead to the determination of the corresponding stability region are similar to the previous ones, and therefore we will be brief in some of the following discussions.

First, under the assumption (H3), $L_2^{(1)}$ is now of the form (see Figure 5)

$$L_2^{(1)} = \{(x,y) \in R^2 : -1 - x - y = 0\}.$$

However, Lemmas 7, 8, 9 and 10 are formally unchanged and can be proved in similar manners.

Next, we observe that the stability region $\Omega(n,m)$ is symmetric with respect to the *origin*, that is,

$$(a,b) \in \Omega(n,m) \Leftrightarrow (-a,-b) \in \Omega(n,m).$$

Indeed, this follows from the fact that when n and m are odd ,

$$P(-\lambda|a,b) = (-\lambda)^n - a(-\lambda)^{n-m} - b = -(\lambda^n - (-a)\lambda^{n-m} - (-b)) = -P(\lambda| - a, -b).$$

Therefore, we only need to characterize $\Omega(n,m)$ in the right half plane. For this purpose, we break the right half plane into six mutually exclusive and exhaustive subregions A, B', C, D and E respectively by (33), (38), (35), (36) and (37). Then as in the previous Section, we may show that B', C and D are parts of $\Omega(n,m)$ but A is not.
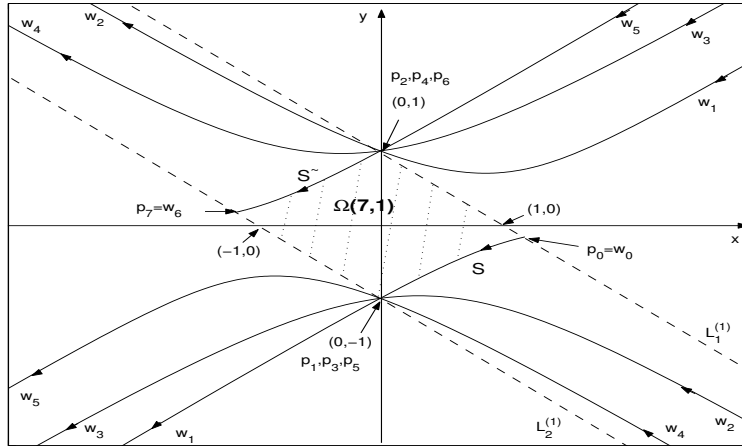

Figure 5: $n = 7, m = 1$

With these preparatory results, we may then prove our second main Theorem by means of the same reasoning used in the proof of Theorem 1 (see Figures 5 and 6).

THEOREM 2. Suppose (H1) and (H3) hold. If $m = 1$, let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval $(0, \pi/n)$. If $m \geq 3$, let

$$J = \begin{cases} (p_{i^*}, q_{j^*}) & \text{if } i^* \text{ is odd} \\ (q_{j^*}, p_{i*+1}) & \text{if } i^* \text{ is even} \end{cases},$$
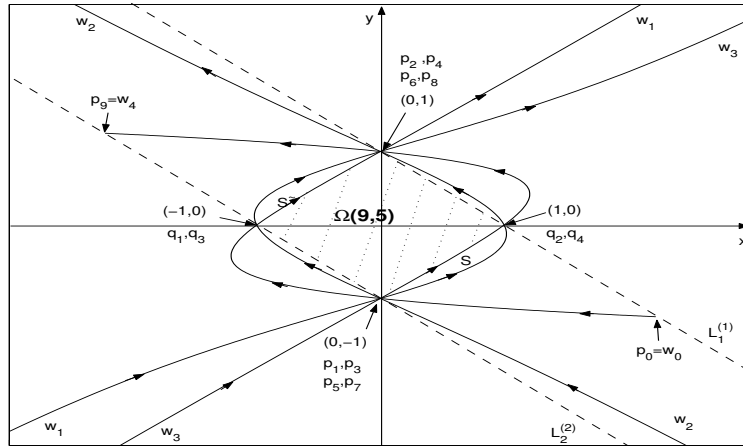
Figure 6: $n = 9, m = 5$

where $j^* \in \{2, 4, ..., m - 1\}$ is determined by (32) and p_{i^*}, p_{i^*+1} are points in P that are closest to q_{j^*}, and let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval J. Let $\widetilde{S}$ be the curve which is the symmetric image of S with respect to the origin. Then $\Omega(n, m)$ is the set of points in the interior of the (bounded) region bounded by $L_1^{(1)}$, $L_2^{(1)}$, S and $\widetilde{S}$.

7   Exact Stability Region under (H4)

In view of (3), it is suspected that the assumption (H4) is 'symmetric' to the assumption (H1) and hence the corresponding stability region can be obtained from Theorem 1. In spite of this observation, the assumption $m \leq n$ is not symmetric and hence we are forced to go through the parallel development briefly.

First note that under the assumption (H4), $L_2^{(1)}$ is now of the form (see Figure 7)

$$L_2^{(1)} = \{(x, y) \in R^2 : -1 + x - y = 0\}.$$

Lemma 7 is replaced with the following.

LEMMA 7". Assume (H1) holds. Let $F(t) = (x_1(t), y_1(t))$ be the parametric function defined by (15). Let $I = (\xi_k, \xi_{k+1})$ be one of the constituent open intervals of $[0, \pi] \backslash W$.

(i) If $I = (w_v, p_i)$, where $i < n$ is even and $v > 0$ is even, then the restriction $F|_I$ traces out a (directed) curve in the interior of the second quadrant with 'initial point $(-\infty, +\infty)$' and endpoint $(0, 1)$.

(ii) If $I = (p_i, w_v)$, where $i > 0$ is even and $v < n - m$ is even, then the restriction $F|_I$ traces out a (directed) curve in the interior of the second quadrant with 'initial point $(0, 1)$ and 'endpoint $(-\infty, \infty)$'.

(iii) If $I = (p_i, q_j)$, where $i > 0$ is even and $j < m$ is odd, then the restriction $F|_I$ traces out a (directed) curve in the interior of the second quadrant with initial point $(0, 1)$ and endpoint $(-1, 0)$.

(iv) If $I = (q_j, p_i)$, where $i < n$ is even and $j > 0$ is odd, then the restriction $F|_I$ traces out a (directed) curve in the interior of the second quadrant with initial point $(-1, 0)$ and endpoint $(0, 1)$.

(vi) If $I = (p_{n-1}, \pi)$, then the restriction $F|_I$ traces out a (directed) curve in the interior of the first quadrant with initial point $(0, 1)$ and endpoint $(\frac{n}{n-m}, \frac{m}{n-m})$ under (H4).

An analog of Lemma 8 is not needed since $m > 1$, but an analog of Lemma 9 is. Before stating it formally, we need to determine the curve that plays the same role of S in Lemma 9. To this end, assume (H1) and (H4) hold. Then for each q_j where $j \in \{1, 3, ..., m - 1\}$, by Lemma 2, there is $v \in \{0, 1, ..., n - m - 1\}$ such that $q_j \in I_v$. We consider the slopes of the graphs of the parametric function $(x_1(t), y_1(t))$ at $t = q_j$. First note that

$$\frac{dx_1}{dy_1}(t) = \frac{m \sin(nt) \cos(nt - mt) + n \sin(mt)}{n \sin(mt) \cos(nt - mt) - m \sin(nt)}.$$

Since $\gcd(n, m) = 1$, we see that $\sin(nq_j) \neq 0$ for $j \in \{1, 3, ..., m - 1\}$ and

$$\frac{dx_1}{dy_1}(q_j) = \cos(nq_j) \text{ for } j = 1, 3, ..., m - 1.$$

We assert that $\cos(nq_1), ..., \cos(nq_{m-1})$ are mutually distinct. Indeed, if there are distinct j_1 and j_2 in $\{1, 3, ..., m - 1\}$ such that $\cos(nq_{j_1}) = \cos(nq_{j_2})$, then $nj_1 = \pm nj_2 \bmod m$. Since $\gcd(n, m) = 1$, then $j_1 = \pm j_2 \bmod m$. If $j_1 = j_2 \bmod m$, then since $0 < j_1, j_2 < m$, we have $j_1 = j_2$, which is a contradiction. If $j_1 = -j_2 \bmod m$, then since $0 < j_1, j_2 < m$, we have $j_1 = m - j_2$, then $\cos(nq_{j_1}) = -\cos(nq_{j_2})$ since n is odd. So $\cos(nq_{j_1}) = \cos(nq_{j_2}) = 0$. Thus, $nq_{j_1} = \frac{2k_1+1}{2}\pi$ and $nq_{j_2} = \frac{2k_2+1}{2}\pi$ for some $k_1, k_2 \in \{0, 1, 2, ...\}$. Since $\gcd(n, m) = 1$, then $j_1 = \frac{2k_1'+1}{2}m$ and $j_2 = \frac{2k_2'+1}{2}m$ for some $k_1', k_2' \in \{0, 1, 2, ...\}$. Since $0 < j_1, j_2 < m$, then $k_1' = k_2' = 0$ and $j_1 = j_2$, which is a contradiction. Among the derivatives $\frac{dx_1}{dy_1}(q_1), \frac{dx_1}{dy_1}(q_3), ..., \frac{dx_1}{dy_1}(q_{m-1})$, we let $q_{j^*} \in Q$ where $j^* \in \{1, 3, ..., m - 1\}$ such that

$$\frac{dx_1}{dy_1}(q_{j^*}) = \max \left\{ \frac{dx_1}{dy_1}(q_1), \frac{dx_1}{dy_1}(q_3), ..., \frac{dx_1}{dy_1}(q_{m-1}) \right\}. \tag{39}$$

Since $q_{j^*} = \xi_k$ for some $\xi_k \in P \cup Q \cup W$ and $j^* \in \{1, 3, ..., m - 1\}$, by Lemma 2, there are p_{i^*} and p_{i^*+1} in P which are closest to q_{j^*} and $p_{i^*} = \xi_{k-1} < \xi_k < \xi_{k+1} = p_{i^*+1}$. If i^* is even, by Lemma 5 and Lemma 7", the parametric function $(x_1(t), y_1(t))$ over the interval (ξ_{k-1}, ξ_k) traces out a (directed) curve in the interior of the second quadrant with initial point $(0, 1)$ and endpoint $(-1, 0)$. Similarly, if i^* is odd, then the parametric function $(x_1(t), y_1(t))$ over the interval (ξ_k, ξ_{k+1}) traces out a (directed) curve in the interior of the second quadrant of the plane with initial point $(-1, 0)$ and endpoint $(0, 1)$. We will now let J be either (ξ_{k-1}, ξ_k) or (ξ_k, ξ_{k+1}) defined above. We will also let Ω_* be the set of points strictly inside the second quadrant and bounded by the x-axis, y-axis and the curve traced out by $(x_1(t), y_1(t))$ over the interval J.

LEMMA 9". Let

$$J = \begin{cases} (p_{i^*}, q_{j^*}) & \text{if } i^* \text{ is even} \\ (q_{j^*}, p_{i^*+1}) & \text{if } i^* \text{ is odd} \end{cases},$$

where $j^* \in \{1, 3, ..., m-1\}$ is determined by (39) and p_{i^*}, p_{i^*+1} are points in P that are closest to q_{j^*}, and let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval J. Then the set of points Ω_* in the interior of the second quadrant bounded by the x-axis, the y-axis and the curve S cannot contain any points of $L_1^{(1)}$, $L_2^{(1)}$, $S_0^{(1)}$, $S_1^{(1)}, ..., S_{n-m-2}^{(1)}$ or $S_{n-m-1}^{(1)}$.

The proof of the above result is similar to that of Lemma 9 and hence is omitted.

Lemma 10 is formally unchanged if the assumption (H2) is replaced by (H4).

Next, we observe that the stability region $\Omega(n, m)$ is symmetric with respect to the x-axis, that is,

$$(a, b) \in \Omega(n, m) \Leftrightarrow (a, -b) \in \Omega(n, m).$$

Indeed, this follows from the fact that when n and m are odd ,

$$P(-\lambda|a, b) = (-\lambda)^n - a(-\lambda)^{n-m} - b = -\lambda^n + a\lambda^{n-m} - b = -P(\lambda|a, -b).$$

Therefore, we only need to characterize $\Omega(n, m)$ in the upper half plane. For this purpose, we break the up half plane into five mutually exclusive and exhaustive subregions:

$$A'' = \{(x, y) \in R^2 : y \geq 1 - x, \ y \geq 0\},$$

$$B'' = \{(0, y) \in R^2 : 0 < y < 1\},$$

$$C'' = \{(x, 0) \in R^2 : x < 1\},$$

$$D'' = \{(x, y) \in R^2 : y < 1 - x, \ x > 0, \ y > 0\},$$

and

$$E'' = \{(x, y) \in R^2 : y < 1 - x, \ x < 0, \ y > 0\}.$$

Let

$$\tilde{C}'' = \{(x, 0) \in R^2 : |x| < 1\}$$

be a subset of C''. Then B'', $\tilde{C}''$ and D'' are parts of $\Omega(n, m)$ but A'' is in the complement of $\Omega(n, m)$.

Under the above preparatory results (together with Lemma 4'), the following main result is proved in a manner similar to that of Theorem 1.

THEOREM 3. Suppose (H1) and (H4) hold. Let

$$J = \begin{cases} (p_{i^*}, q_{j^*}) & \text{if } i^* \text{ is even} \\ (q_{j^*}, p_{i^*+1}) & \text{if } i^* \text{ is odd} \end{cases},$$

where $j^* \in \{1, 3, ..., m-1\}$ is determined by (39) and p_{i^*}, p_{i^*+1} are points in P that are closest to q_{j^*}, and let S be the curve of the parametric function $(x_1(t), y_1(t))$ over the interval J. Let $\tilde{S}$ be the curve which is the symmetric image of S with respect to the x-axis. Then $\Omega(n, m)$ is the set of points in the interior of the (bounded) region bounded by $L_1^{(1)}$, $L_2^{(1)}$ and the curve $S \cup \{(0, 0)\} \cup \tilde{S}$.
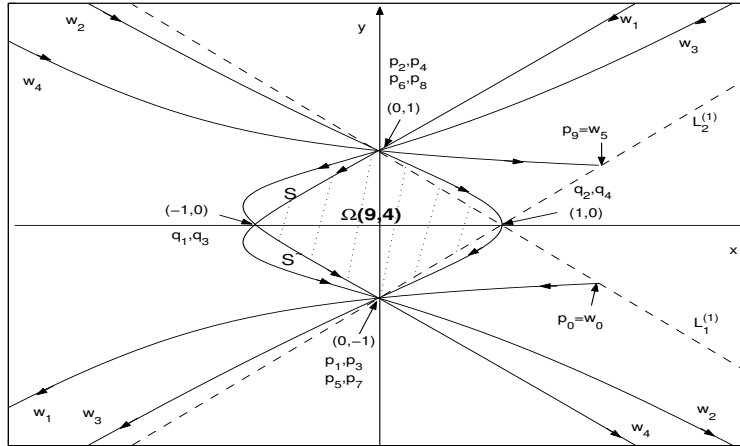
Figure 7: $n = 9, m = 4$

For illustration, let us consider the case where $n = 9$ and $m = 4$. Since $q_j = j\pi/4$ for $j = 0, 1, 2, 3, 4$, we see that

$$\begin{aligned} \cos(9q_1) &= 0.7071... \\ \cos(9q_3) &= -0.7071... \end{aligned}$$

so that $j^* = 1$. In view of (5), we see that p_2 and p_3 are closest to q_1 and $p_2 < q_1 < p_3$. Since 2 is even, we see that $J = (p_2, q_1) = (2\pi/9, \pi/4)$. Thus

$$S = \{(x_1(\theta), y_1(\theta)) : \theta \in J\}$$

which together with $\tilde{S}$ and the stability region $\Omega(9, 4)$ are depicted in Figure 7.

As our final remark, let us also consider the case where $n = 7$ and $m = 6$. Since $q_j = j\pi/6$ for $j = 0, 1, 2, 3, 4, 5, 6$, we see that

$$\begin{aligned} \cos(7q_1) &= -0.8660... \\ \cos(7q_3) &= 0 \\ \cos(7q_5) &= 0.8660... \end{aligned}$$

so that $j^* = 5$. In view of (5), we see that p_5 and p_6 are closest to q_5 and $p_5 < q_5 < p_6$. It is of interest to note that $\cos(7q_3) = 0$ so that we use dx_1/dy_1 in (39) instead of dy_1/dx_1 to avoid a minor technicality.

8   Remarks

The interval J in Theorems 1, 2 and 3 are not difficult to construct by means of a simple algorithm. For the sake of convenience, we present the output from this program in three self explanatory tables depicted in Figures 8, 9 and 10.

m \ n	2	4	6	8	10	12	14	16	18
1	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)
3		(q₂,p₃)		(p₅,q₂)	(q₂,p₇)		(p₉,q₂)	(q₂,p₁₁)	
5			(q₄,p₅)	(q₄,p₇)		(p₉,q₄)	(p₁₁,q₄)	(q₄,p₁₃)	(q₄,p₁₅)
7				(q₆,p₇)	(q₂,p₃)	(q₆,p₁₁)		(p₁₃,q₆)	(p₅,q₂)
9					(q₈,p₉)		(q₈,p₁₃)	(q₈,p₁₅)	
11						(q₁₀,p₁₁)	(p₅,q₄)	(q₂,p₃)	(q₁₀,p₁₄)
13							(q₁₂,p₁₃)	(q₄,p₅)	(p₁₁,q₈)
15								(q₁₄,p₁₅)	
17									(q₁₆,p₁₇)

Figure 8: H2, $p_i = i\pi/n$, $q_j = j\pi/m$

m \ n	3	5	7	9	11	13	15	17	19
1	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)	(0,p₁)
3		(p₃,q₂)	(q₂,p₅)		(p₇,q₂)	(q₂,p₉)		(p₁₁,q₂)	(q₂p₁₃)
5			(p₅,q₄)	(p₄,q₄)	(q₄,p₉)	(q₄,p₁₁)		(p₁₃,q₄)	(p₁₅,q₄)
7				(p₇,q₆)	(p₃,q₂)	(p₁₁,q₆)	(q₆,p₁₃)	(q₂,p₅)	(p₆,q₁₇)
9					(p₉,q₈)	(p₁₁,q₈)		(p₁₅,q₈)	(q₈,p₁₇)
11						(p₁₁,q₁₀)	(p₁₃,q₁₀)	(p₃,q₂)	(q₄,p₇)
13							(p₁₃,q₁₂)	(p₁₅,q₁₂)	(q₁₇,p₁₂)
15								(p₁₅,q₁₄)	(p₁₇,q₁₄)
17									(q₁₇,p₁₆)

Figure 9: H3, $p_i = i\pi/n$, $q_j = j\pi/m$

A quick look at these three tables reveal certain regular patterns, some of which are listed as follows:

- If ((H1) holds and) $m = n - 1$, then $J = (q_{m-1}, p_{n-1})$.

- If (H1) and (H3) hold, $n > 5$ and $m = n - 2$, then $J = (p_{n-2}, q_{m-1})$.

- If (H1) and (H4) hold and $m = 2$, then

$$J = \begin{cases} (p_1, q_{2k}) & \text{if } n = 4k - 1 \text{ for some integer } k \\ (q_{2k}, p_1) & \text{if } n = 4k + 1 \text{ for some integer } k \end{cases} .$$

- If (H1) and (H4) hold and $m = 4$, then

$$J = \begin{cases} (p_k, q_1) & \text{if } n = 4k + 1 \text{ for some even integer } k \\ (q_3, p_{3k+1}) & \text{if } n = 4k + 1 \text{ for some odd integer } k \\ (q_1, p_k) & \text{if } n = 4k - 1 \text{ for some even integer } k \\ (p_{3k-1}, q_3) & \text{if } n = 4k - 1 \text{ for some odd integer } k \end{cases} .$$

m \ n	3	5	7	9	11	13	15	17	19
2	$(q_1 p_2)$	(p_2, q_1)	(q_1, p_4)	(p_4, q_1)	(q_1, p_6)	(p_6, q_1)	(q_1, p_8)	(p_8, q_1)	(q_1, p_{10})
4		(q_3, p_4)	(q_1, p_2)	(p_2, q_1)	(p_8, q_3)	(q_3, p_{10})	(q_1, p_4)	(p_4, q_1)	$(p_{14} q_3)$
6			(q_5, p_6)		(q_1, p_2)	(p_2, q_1)		(p_{14}, q_5)	(q_5, p_{16})
8				(q_7, p_8)	(p_4, q_3)	(p_8, q_5)	(q_1, p_2)	(p_2, q_1)	(q_5, p_{12})
10					(q_9, p_{10})	(q_3, p_4)		(q_7, p_{12})	(q_1, p_2)
12						(q_{11}, p_{12})		(q_7, p_{10})	(q_5, p_8)
14							(q_{13}, p_{12})	(p_6, q_5)	(p_4, q_3)
16								(q_{15}, p_{16})	(q_5, p_6)
18									(q_{17}, p_{18})

Figure 10: H4, $p_i = i\pi/n$, $q_j = j\pi/m$

- If (H1) holds and $m = 3$, then

$$J = \begin{cases} (q_2, p_{2(\frac{n-1}{3})+1}) & \text{if } n = 1 \bmod 3 \\ (p_{2(\frac{n-2}{3})+1}, q_2) & \text{if } n = 2 \bmod 3 \end{cases}.$$

These observations can be proved quite easily. For instance, suppose (H1) holds and $m = n - 1$. If n is even, then $J = (0, p_1) = (q_{m-1}, p_{n-1})$. If $n > 2$, then

$$\frac{dy_1}{dx_1}(q_j) = \frac{-1}{\cos j\pi/(n-1)}, \ j = 2, 4, ..., m - 1.$$

Since $\frac{2\pi}{n-1} < \frac{4\pi}{n-1} < \cdots < \frac{(n-2)\pi}{n-1} < \pi$ and $\pi/2 < (n-2)\pi/(n-1)$ for $n > 3$. Hence $\Gamma \neq \emptyset$ and

$$\frac{dy_1}{dx_1}(q_{n-2}) = \min \Gamma.$$

Therefore by the definition of J, we pick $J = (q_{n-2}, p_{n-1}) = (q_{m-1}, p_{n-1})$.

The other observations can similarly be proved.

We have assumed that n and m are positive integers. However, the equation

$$f_k = a f_{k-m} + b f_{k-n}$$

still makes sense if n or m are negative integers. For instance, in the equation

$$f_k = a f_{k+1} + b f_{k+2},$$

f_k may be interpreted as the expected net present value (NPV) in the time period k of a cooperation and it is speculated that it depends on the respective expected NPV of the coming two time periods. The same question then arises as whether f_k will tend to 0 or not. This question can be answered by our previous results. Indeed, we may assume without loss of generality that $\bar{n}$ and $\bar{m}$ are integers such that $\gcd(\bar{n}, \bar{m}) = 1$. The pair (a, b) is said to be a point of stability (for the equation)

$$f_k = a f_{k-\bar{m}} + b f_{k-\bar{n}}, \tag{40}$$

if all solutions tend to 0. Let $\Omega(n, m)$ be the region of stability of (1) found in Theorems 1, 2 and 3.

The case where $\bar{n} > \bar{m} > 0$ has already been handled. For the case where $\bar{m} > \bar{n} > 0$, since the corresponding characteristic polynomial is $\lambda^{\bar{m}} - b\lambda^{\bar{m}-\bar{n}} - a = 0$, we see that (a, b) is a point of stability (for (40)) if, and only if, $(a, b) \in \Omega(\bar{m}, \bar{n})$.

Suppose $\bar{n} < \bar{m} < 0$. If $(a, b) = (0, 0)$, then (40) takes the form $f_k = 0$. Thus the point $(a, b) = (0, 0)$ is clearly a point of stability. If $b = 0$ and $a \neq 0$, then (40) can be written in the form $f_i = \frac{1}{a} f_{i+\bar{m}}$ by letting $i = k - \bar{m}$. Thus (a, b) is a point of stability if, and only if, $(\frac{1}{a}, 0) \in \Omega(-\bar{n}, -\bar{m})$. If $b \neq 0$, let $i = k - \bar{n}$, then (40) can be written in the form $f_i = \frac{1}{b} f_{i+\bar{n}} - \frac{a}{b} f_{i+(\bar{n}-\bar{m})}$. Thus (a, b) is a point of stability if, and only if, $(-\frac{a}{b}, \frac{1}{b}) \in \Omega(-\bar{n}, \bar{m} - \bar{n})$.

Suppose $\bar{m} < \bar{n} < 0$. The point $(a, b) = (0, 0)$ is a point of stability. If $a = 0$ and $b \neq 0$, then (a, b) is a point of stability if, and only if, $(0, \frac{1}{b}) \in \Omega(-\bar{m}, -\bar{n},)$. If $b \neq 0$, then (a, b) is a point of stability if, and only if, $(-\frac{b}{a}, \frac{1}{a}) \in \Omega(-\bar{m}, \bar{n} - \bar{m})$.

Suppose $\bar{n} > 0 > \bar{m}$. The point $(a, b) = (0, 0)$ is a point of stability. If $a = 0$ and $b \neq 0$, then (a, b) is a point of stability if, and only if,

$$
\begin{cases}
(b, 0) \in \Omega(-\bar{m}, \bar{n}) & \text{if } -\bar{m} > \bar{n} \\
(0, b) \in \Omega(\overline{n}, -\bar{m}) & \text{if } -\bar{m} < \bar{n}
\end{cases}.
$$

If $a \neq 0$, then (a, b) is a point of stability if, and only if, $(\frac{1}{a}, -\frac{b}{a}) \in \Omega(\bar{n} - \bar{m}, -\bar{m})$.

Suppose $\bar{m} > 0 > \bar{n}$. The point $(a, b) = (0, 0)$ is a point of stability. If $b = 0$ and $a \neq 0$, then (a, b) is a point of stability if, and only if,

$$
\begin{cases}
(a, 0) \in \Omega(\bar{m}, -\bar{n}) & \text{if } \bar{m} > -\bar{n} \\
(0, a) \in \Omega(-\overline{n}, \overline{m}) & \text{if } \bar{m} < -\bar{n}
\end{cases}.
$$

If $b \neq 0$, then (a, b) is a point of stability if, and only if, $(\frac{1}{b}, -\frac{a}{b}) \in \Omega(\bar{m} - \bar{n}, -\bar{n})$.

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