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Extremal $\mathcal{I}$-Limit Points Of Double Sequences*

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Abstract

The concept of ideal convergence was introduced by P. Kostyrko et al in 2000. This concept was extended to the double sequences by B.C. Tripathy in 2005. Throughout this paper we will present multidimensional ideal analogues of the results presented by J.A. Fridy and C. Orhan in 1997. To achieve this goal, multidimensional ideal analogues of the definitions for $\mathcal{I}$-bounded sequences, $\mathcal{I}$-inferior and $\mathcal{I}$-superior will be presented.

1 Introduction and Background

The notion of ideal convergence was introduced first by P. Kostyrko et al [7] as an interesting generalization of statistical convergence [1],[11].

The concept of a double sequence was initially introduced by Pringsheim [9] in the 1900s. Since then, this concept has been studied by many authors, [5],[10],[12],[13].

The purpose of this paper is to present natural definitions of the concepts of ideal limit superior and inferior of double sequences and develop some ideal analogues of properties of the ordinary limit superior and inferior of double sequences.

The notion of statistical convergence depends on the density of the subsets of $\mathbb{N}$, the set of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\chi_E(k)$ exists [2]. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$. In this case it is denoted as st-$\lim x_n = L$.

A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (**i**) $\emptyset \in \mathcal{I}$; (**ii**) for each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$; (**iii**) for each $A \in \mathcal{I}$ and each $B \subseteq A$, we have $B \in \mathcal{I}$. If the ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ each $x \in Y$, then it is an admissible ideal [7],[8].

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal (i.e., $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$) in $\mathbb{N}$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $\xi \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [6],[7].

A double sequence $x = (x_{nk})$ is said to converge in Pringsheim's sense if there exists a real number L such that (x_{nk}) converges to L as both n and k tend to infinity independently of one another; in this case we write P-$\lim_{n,k\to\infty} x_{nk} = L$. It is clear that

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the convergence of (x_{nk}) in Pringsheim's sense does not guarantee the boundedness of (x_{nk}) [9].

2 Preliminaries

The notion of statistically convergent double sequence was introduced by B.C. Tripathy [12]. It depends on the density of subsets of $\mathbb{N}\times\mathbb{N}$. A subset E of $\mathbb{N}\times\mathbb{N}$ (introduced by B.C. Tripathy [12]) is said to have density $\rho(E)=\lim_{p,q\to\infty}\frac{1}{pq}\sum_{n\le p}\sum_{k\le q}\chi_E(n,k)$ whenever this limit exists.

A double sequence (x_{nk}) is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon>0$, $\rho(\{(n,k)\in\mathbb{N}\times\mathbb{N}:|x_{nk}-L|\ge\varepsilon\})=0$ [13].

This concept was extended to $\mathcal{I}$-convergence of double sequences by B.C. Tripathy in [13]. In order to distinguish between the ideals of $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ we shall denote the ideals of $2^{\mathbb{N}}$ by $\mathcal{I}$ and that of $2^{\mathbb{N}\times\mathbb{N}}$ by $\mathcal{I}_2$, respectively. In general, there is no connection between $\mathcal{I}$ and $\mathcal{I}_2$.

Let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$. Then a double sequence (x_{nk}) is said to be $\mathcal{I}$-convergent to L in Pringsheim's sense if for every $\varepsilon>0$,

$$\{(n,k)\in\mathbb{N}\times\mathbb{N}:|x_{nk}-L|\ge\varepsilon\}\in\mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2\text{-}\lim x_{nk}=L$ [13].

In what follows, we will give some examples of ideals and corresponding $\mathcal{I}_2$-convergen cies.

(I) Let $\mathcal{I}_2(f)$ be the family of all finite subsets of $\mathbb{N}\times\mathbb{N}$. Then $\mathcal{I}_2(f)$ is an admissible ideal in $\mathbb{N}\times\mathbb{N}$ and $\mathcal{I}_2(f)$ convergence coincides with the convergence in Pringsheim's sense [9].

(II) Let $A\subseteq\mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $A(n,k)$ be the number of (i,j) in A such that $i\le n$ and $j\le k$. In this case if the sequence $(A(n,k)/nk)$ has a limit in Pringsheim's sense then we say that A has a double natural density and it is defined as $\lim_{n,k}(A(n,k)/nk)=\rho(A)$. Put $\mathcal{I}_2(\rho)=\{A\subset\mathbb{N}\times\mathbb{N}:\rho(A)=0\}$. Then $\mathcal{I}_2(\rho)$ is an admissible ideal in $\mathbb{N}\times\mathbb{N}$ and $\mathcal{I}_2(\rho)$ convergence coincides with the statistical convergence in Pringsheim's sense [12].

Below is an example of $\mathcal{I}$-convergence of double sequences in Pringsheim's sense.

EXAMPLE 1. Let $\mathcal{I}=\mathcal{I}_2(\rho)$. Define the double sequence (x_{nk}) by

$$x_{nk}=\begin{cases}1 & \text{, if } n,k\in\mathbb{N} \text{ and } n,k \text{ are squares}\\ 0 & \text{, otherwise.}\end{cases}$$

and let $L=0$. Then for every $\varepsilon>0$

$$\rho(\{(n,k)\in\mathbb{N}\times\mathbb{N}:|x_{nk}-L|\ge\varepsilon\})=\rho(A)=\lim_{n,k}\frac{A(n,k)}{nk}\le\lim_{n,k}\frac{\sqrt{n}\sqrt{k}}{nk}=0,$$

i.e., the set A has double natural density zero for every $\varepsilon>0$. This implies that $st\text{-}\lim_{n,k\to\infty}\lim|x_{nk}-L|=0$ in Pringsheim's sense. But the sequence (x_{nk}) is not convergent to L in Pringsheim's sense.

A double sequence (x_{nk}) which is $\mathcal{I}$-convergent to zero in Pringsheim's sense is called an $\mathcal{I}$-null double sequence in Pringsheim's sense [13].

A double sequence (x_{nk}) is said to be $\mathcal{I}$-Cauchy if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that $\{(n,k) \in \mathbb{N} \times \mathbb{N} : |x_{nk} - x_{st}| \geq \varepsilon\} \in \mathcal{I}_2$ [13].

3 Ideal Boundedness of Double Sequences

In the paper [3] the notions of statistical limit point and statistical cluster point were introduced. In [4], the authors introduced the notions of extremal statistical limit points (statistical $\liminf x$, statistical $\limsup x$). In the paper [7] the notions of $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point of a sequence of elements of a metric space were introduced. These notions generalize the notions of statistical limit point and statistical cluster point.

Recall that a number ξ is said to be an $\mathcal{I}$-limit point of $x = (x_n)$ provided that there is a set $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{n_k} = \xi$. A number ξ is said to be an $\mathcal{I}$-cluster point of $x = (x_n)$ if for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_n - \xi| < \varepsilon\} \notin \mathcal{I}$.

From now on, unless otherwise expressed we shall deal with an admissible ideals of $2^{\mathbb{N}\times\mathbb{N}}$ and the notations mentioned above.

REMARK 1. Note that for any set $M \subsetneqq \mathbb{N}$ at least one of the statements $M \in \mathcal{I}$ and $\mathbb{N} \backslash M \in \mathcal{I}$ does not hold.

Further, we will give a generalization of the notions of statistical $\liminf x$ and statistical $\limsup x$ of [4] for a double sequence $x = (x_{nk})$.

DEFINITION 1. Let $\mathcal{I}_2$ be a nontrivial ideal of $2^{\mathbb{N}\times\mathbb{N}}$. A number ξ is said to be an $\mathcal{I}$-limit point of the double sequence x_{nk} in Pringsheim's sense provided that there exists a set $M = \{n_1 < n_2 < ...\} \times \{k_1 < k_2 < ...\} \subset \mathbb{N} \times \mathbb{N}$ such that $M \notin \mathcal{I}_2$ and $P\text{-}\lim_{i,j\to\infty} x_{n_i k_j} = \xi$.

DEFINITION 2. Let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$. A number ζ is said to be an $\mathcal{I}$-cluster point of the double sequence x_{nk} in Pringsheim's sense if for each $\varepsilon > 0$,

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : |x_{nk} - \zeta| < \varepsilon\} \notin \mathcal{I}_2.$$

We write

$$M_t = \{(n,k) : x_{nk} > t\} \text{ and } M^t = \{(n,k) : x_{nk} < t\}, \text{ for } t \in \mathbb{R}.$$

DEFINITION 3. a) If there is a $t \in \mathbb{R}$ such that $M_t \notin \mathcal{I}_2$, we define

$$\mathcal{I} - \limsup x = \sup\{t \in \mathbb{R} : M_t \notin \mathcal{I}_2\}.$$

If $M_t \in \mathcal{I}_2$ holds for each $t \in \mathbb{R}$, then we define $\mathcal{I}$-$\limsup x = -\infty$.

b) If there is a $t \in \mathbb{R}$ such that $M^t \notin \mathcal{I}_2$, we define

$$\mathcal{I} - \liminf x = \inf\left\{t \in \mathbb{R} : M^t \notin \mathcal{I}_2\right\}.$$

If $M^t \in \mathcal{I}_2$ holds for each $t \in \mathbb{R}$, then we define $\mathcal{I}$-$\liminf x = +\infty$.

DEFINITION 4. Let $\mathcal{I}_2$ be an admissible ideal of $2^{\mathbb{N}\times\mathbb{N}}$. A real double sequence x_{nk} is said to be $\mathcal{I}$-bounded if there is a $K>0$ such that $\{(n,k)\in\mathbb{N}\times\mathbb{N}:|x_{nk}|>K\}\in\mathcal{I}_2$.

A simple example will help to illustrate the concepts just defined above.

EXAMPLE 2. If we define x_{nk} by

$$x_{nk}=\begin{cases} k & , \quad k \text{ is an odd square} \\ 2 & , \quad k \text{ is an even square} \\ 1 & , \quad k \text{ is an odd nonsquare} \\ 0 & , \quad k \text{ is an even nonsquare} \end{cases}$$

or by

$$x_{nk}=\begin{cases} n & , \quad n \text{ is an odd square} \\ 2 & , \quad n \text{ is an even square} \\ 1 & , \quad n \text{ is an odd nonsquare} \\ 0 & , \quad n \text{ is an even nonsquare} \end{cases},$$

then x_{nk} is not bounded from above but it is $\mathcal{I}$-bounded. We have $\{t\in\mathbb{R}:M_t\notin\mathcal{I}_2\}=(-\infty,1)$, $\{t\in\mathbb{R}:M^t\notin\mathcal{I}_2\}=(0,\infty)$; $\mathcal{I}_2$-$\limsup x_{nk}=1$, $\mathcal{I}_2$-$\liminf x_{nk}=0$. On the other hand, x_{nk} cannot be $\mathcal{I}$-convergent in Pringsheim's sense and the set of $\mathcal{I}$-cluster points in Pringsheim's sense is $\{0,1\}$. So we have the following remark.

REMARK 2. If $\mathcal{I}_2=\mathcal{I}_2(f)$, then the above Definition 1 yields the usual definition of P-$\limsup_{n,k\to\infty}x_{nk}$ and P-$\liminf_{n,k\to\infty}x_{nk}$.

The next statement is an analogue of Theorem 1.2 of [4].

THEOREM 1. (i) $\beta=\mathcal{I}_2$-$\limsup x_{nk}\Longleftrightarrow$ For each $\varepsilon>0$,

$$\{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}>\beta-\varepsilon\}\notin\mathcal{I}_2 \text{ and } \{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}>\beta+\varepsilon\}\in\mathcal{I}_2.$$

(ii) $\alpha=\mathcal{I}_2$-$\liminf x_{nk}\Longleftrightarrow$ For each $\varepsilon>0$,

$$\{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}<\alpha+\varepsilon\}\notin\mathcal{I}_2 \text{ and } \{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}<\alpha-\varepsilon\}\in\mathcal{I}_2.$$

PROOF. (i) We prove the necessity first. Let $\varepsilon>0$ be given. Since $\beta+\varepsilon>\beta$, we have $(\beta+\varepsilon)\notin\{t:M_t\notin\mathcal{I}_2\}$ and $\{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}>\beta+\varepsilon\}\in\mathcal{I}_2$. Similarly, since $\beta-\varepsilon<\beta$, there exists some t' such that $\beta-\varepsilon<t'<\beta$ and $t'\in\{t:M_t\notin\mathcal{I}_2\}$. Thus $\{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}>t'\}\notin\mathcal{I}_2$ and $\{(n,k)\in\mathbb{N}\times\mathbb{N}:x_{nk}>\beta-\varepsilon\}\notin\mathcal{I}_2$.

Now let us prove the sufficiency. If $\varepsilon>0$ then $(\beta+\varepsilon)\notin\{t:M_t\notin\mathcal{I}_2\}$ and $\mathcal{I}_2$-$\limsup x_{nk}\le\beta+\varepsilon$. On the other hand, we already have $\mathcal{I}_2$-$\limsup x_{nk}\ge\beta-\varepsilon$, and this means that $\mathcal{I}_2$-$\limsup x_{nk}=\beta$, as desired.

(ii) Similarly as in (i).

By Definition 2 we see that Theorem 1 can be interpreted by saying that $\mathcal{I}_2$-$\limsup x_{nk}$ and $\mathcal{I}_2$-$\liminf x_{nk}$ are the greatest and the least $\mathcal{I}$-cluster points of (x_{nk}) in Pringsheim's sense. The next theorem reinforces this observation.

THEOREM 2. For every real double sequence x_{nk},

$$\mathcal{I}_2\text{-}\liminf x_{nk}\le\mathcal{I}_2\text{-}\limsup x_{nk}.$$

PROOF. If x_{nk} is any real double sequence then we have three possibilities:
(1) $\mathcal{I}_2$-$\limsup x_{nk} = +\infty$. In this case there is nothing to prove.
(2) $\mathcal{I}_2$-$\limsup x_{nk} = -\infty$. If this is the case, then we have

$$t \in \mathbb{R} \Rightarrow M_t \in \mathcal{I}_2$$

and

$$t \in \mathbb{R} \Rightarrow M^t \notin \mathcal{I}_2.$$

Thus, $\mathcal{I}_2$-$\liminf x_{nk} = \inf \{t : M^t \notin \mathcal{I}_2\} = \inf \mathbb{R} = -\infty$ and $\mathcal{I}_2$-$\liminf x_{nk} \leq \mathcal{I}_2$-$\limsup x_{nk}$.

(3) $-\infty < \mathcal{I}_2$-$\limsup x_{nk} < +\infty$. For this case there exists a $\beta \in \mathbb{R}$ such that $\beta = \mathcal{I}_2$-$\limsup x_{nk}$. For any $t \in \mathbb{R}$,

$$\beta < t \Rightarrow M_t \in \mathcal{I}_2 \text{ and } M^t \notin \mathcal{I}_2.$$

But this means that $\mathcal{I}_2$-$\liminf x_{nk} = \inf \{t : M^t \notin \mathcal{I}_2\} \leq \beta$.

THEOREM 3. The inequalities

$$P\text{-}\liminf x_{nk} \leq \mathcal{I}_2\text{-}\liminf x_{nk} \leq \mathcal{I}_2\text{-}\limsup x_{nk} \leq P\text{-}\limsup x_{nk} \tag{1}$$

hold for every real double sequence x_{nk}.

PROOF. The case P-$\limsup x_{nk} = +\infty$ is straightforward. Let P-$\limsup x_{nk} = L < +\infty$. Then for any $t' > L$, we have $M_{t'} \in \mathcal{I}_2$. So, $t' \notin \{t : M_t \notin \mathcal{I}_2\}$ implies that $\mathcal{I}_2$-$\limsup x_{nk} = \sup \{t : M_t \notin \mathcal{I}_2\} < t'$ and $\mathcal{I}_2$-$\limsup x_{nk} \leq L$. This proves the last inequality. As for the first one, if P-$\liminf x_{nk} = -\infty$ then clearly the inequality holds. Let P-$\liminf x_{nk} = T > -\infty$. Then for any $t' < T$, we have $M^{t'} \in \mathcal{I}_2$. So $t' \notin \{t : M^t \notin \mathcal{I}_2\}$ implies that $\mathcal{I}_2$-$\liminf x_{nk} = \sup \{t : M^t \notin \mathcal{I}_2\} > t'$ and $\mathcal{I}_2$-$\limsup x_{nk} \geq T$.

REMARK 3. If $\mathcal{I}_2$-$\lim x_{nk}$ exists, then x_{nk} is $\mathcal{I}$-bounded.

REMARK 4. Note that ideal boundedness of double sequences implies that $\mathcal{I}_2$-$\limsup$ and $\mathcal{I}_2$-$\liminf$ are finite.

Recall that the core of a bounded double sequence x_{nk}, that is, P-*core* (x_{nk}), is the interval $[P - \liminf x_{nk}, P - \limsup x_{nk}] = P\text{-}core\,(x_{nk})$. In analogy to the P-*core* (x_{nk}) we first give a definition of $\mathcal{I}$-core of bounded double sequence x_{nk} as follows.

DEFINITION 5. If x_{nk} is any $\mathcal{I}$-bounded real double sequence, then we define its $\mathcal{I}$-core in Pringsheim's sense by

$$[\mathcal{I}_2\text{-}\liminf x_{nk}, \mathcal{I}_2\text{-}\limsup x_{nk}].$$

We use $\mathcal{I}_2$-*core* (x_{nk}) to denote $\mathcal{I}$-core of double sequence (x_{nk}) in Pringsheim's sense.

The following corollary is clear from (1).

COROLLARY 1. If x_{nk} is any real double sequence, then we have

$$\mathcal{I}_2\text{-}core\,(x_{nk}) \subset P\text{-}core\,(x_{nk}).$$

THEOREM 4. A real double sequence x_{nk} is $\mathcal{I}$-convergent in Pringsheim's sense if and only if $\mathcal{I}_2$-$\liminf x_{nk} = \mathcal{I}_2$-$\limsup x_{nk}$.

PROOF. We prove the necessity first. Let $L = \mathcal{I}_2$-$\lim x_{nk}$. Then

$$\{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} > L+\varepsilon\} \in \mathcal{I}_2 \text{ and } \{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} < L-\varepsilon\} \in \mathcal{I}_2.$$

Then for any $t \geq L+\varepsilon$ and $t' < L-\varepsilon$, the sets M_t and $M^{t'}$ are in $\mathcal{I}_2$. We conclude $\sup\{t : M_t \notin \mathcal{I}_2\} \leq L+\varepsilon$ and $\inf\left\{t' : M^{t'} \notin \mathcal{I}_2\right\} \geq L-\varepsilon$. So we get $L = \mathcal{I}_2$-$\liminf x_{nk} = \mathcal{I}_2$-$\limsup x_{nk}$.

To prove sufficiency, let $\varepsilon > 0$ and $L = \mathcal{I}_2$-$\liminf x_{nk} = \mathcal{I}_2$-$\limsup x_{nk}$. Since

$$\begin{aligned}\{(n,k) \in \mathbb{N}\times\mathbb{N} : |x_{nk}-L| \geq \varepsilon\} \quad \subseteq \quad & \{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} > L+\varepsilon\} \\ & \cup \{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} < L-\varepsilon\},\end{aligned}$$

we conclude that $L = \mathcal{I}_2$-$\lim x_{nk}$.

Note that if x_{nk} is a bounded real double sequence, then we denote the set of all $\mathcal{I}$-cluster points of x_{nk} in Pringsheim's sense by $\mathcal{I}_2\left(\Gamma_{x_{nk}}\right)$.

THEOREM 5. Suppose that x_{nk} is a bounded real double sequence. Then

$$\mathcal{I}_2\text{-}\limsup x_{nk} = \max \mathcal{I}_2\left(\Gamma_{x_{nk}}\right)$$

and

$$\mathcal{I}_2\text{-}\liminf x_{nk} = \min \mathcal{I}_2\left(\Gamma_{x_{nk}}\right).$$

PROOF. Let $\mathcal{I}_2$-$\limsup x_{nk} = L = \sup\{t : \{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} > t\} \notin \mathcal{I}_2\}$. If $L' > L$, then there exists some $\varepsilon > 0$ such that $\{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} > L'-\varepsilon\} \in \mathcal{I}_2$. This means that there exists some $\varepsilon > 0$ such that $\{(n,k) \in \mathbb{N}\times\mathbb{N} : |x_{nk}-L'| < \varepsilon\} \in \mathcal{I}_2$, that is, $L' \notin \left(\Gamma_{x_{nk}}\right)$.

Now, we show that L is in fact an $\mathcal{I}$-cluster point of x_{nk} in Pringsheim's sense. Clearly, for each $\varepsilon > 0$ there exists some $t \in (L-\varepsilon, L+\varepsilon)$ such that $\{(n,k) \in \mathbb{N}\times\mathbb{N} : x_{nk} > t\} \notin \mathcal{I}_2$, and this means $\{(n,k) \in \mathbb{N}\times\mathbb{N} : |x_{nk}-L| < \varepsilon\} \notin \mathcal{I}_2$.

Let $\mathcal{I}_2 = \mathcal{I}_2(\rho)$, where $\mathcal{I}_2(\rho) = \{A \subset \mathbb{N}\times\mathbb{N} : \rho(A) = 0\}$ and $\rho(A)$ is the double natural density of the set $A \subset \mathbb{N}\times\mathbb{N}$. Then all these results imply similar theorems for statistically convergent sequences.

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