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Computation Of Polar Moments Of Inertia With Holditch Type Theorem*

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Abstract

In this paper, during the 1-parameter closed planar homothetic motions, the polar moments of inertia for the orbit curves are expressed in the complex plane and the Holditch type theorem for polar moments of inertia is given. In the case of the homothetic scale $h \equiv 1$ the results given by [3] are obtained as a special case.

1 Introduction

H. R. Müller studied the polar moment of inertia, [3], of the closed orbit curve under the 1-parameter closed planar motions and the area of the region bounded by the projection curve under the 1-parameter closed spatial motions, [5]. Combining these two studies of H. R. Müller, we expressed the polar moment of inertia of the closed projection curve under the spatial kinematics. We obtained the relation between the polar moments of inertia of three collinear points and introduced the Holditch type formula for the polar moments of inertia in spatial motions, [6].

In this paper, we generalize the polar moment of inertia of the closed plane curves to the homothetic motions. We derive the polar moment of inertia of any fixed point in plane by means of the polar moments of inertia of three noncollinear points. Thus, using a triangle instead of a line segment, the results and Holditch-type theorems given by [3] and [6] are extended under homothetic motions.

2 Preliminaries

Let E and E' be moving and fixed complex planes and $\{O; \mathbf{e_1}, \mathbf{e_2}\}$ and $\{O'; \mathbf{e'_1}, \mathbf{e'_2}\}$ be their orthonormal coordinate systems, respectively. If the vector $\vec{O'O}$ is represented by the complex number $\mathbf{u}'$ (Figure 1), the motion defined by the transformation

$$\mathbf{x}' = \mathbf{u}' + h\mathbf{x}e^{i\varphi} \tag{1}$$

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is called *1-parameter planar homothetic (equiform, [1]) motion* and denoted by E/E', where h is the homothetic scale and φ is the rotation angle of the motion E/E', that is, the angle between the vectors $\mathbf{e_1}$ and $\mathbf{e'_1}$, and the complex numbers $\mathbf{x} = x_1 + ix_2$, $\mathbf{x'} = x'_1 + ix'_2$ represent the point $X \in E$ with respect to the moving and the fixed rectangular coordinate systems, respectively. The homothetic scale h, the rotation angle φ, $\mathbf{x}$, $\mathbf{x'}$ and $\mathbf{u'}$ are continuously differentiable functions of a real parameter t. Furthermore, at the initial time $t = 0$ the coordinate systems are coincident.

Let the complex number $\mathbf{u} = u_1 + iu_2$ represents the origin of the fixed system in the moving system. Then, if we take $X' = O'$, we obtain $\mathbf{x'} = \mathbf{0}$ and $h\mathbf{x} = \mathbf{u}$. Thus, we have from Eq.(1)

$$\mathbf{u'} = -\mathbf{u}e^{i\varphi}. \tag{2}$$

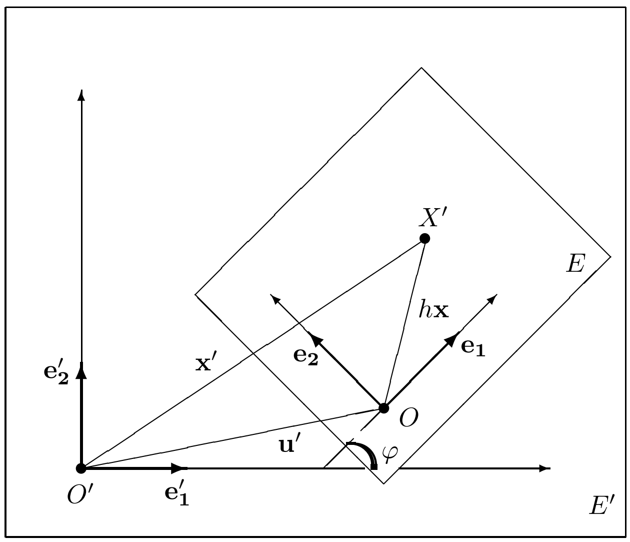

Figure 1. The moving and the fixed planes

During the 1-parameter planar homothetic motion if there exists a number $T > 0$ such that

$$\left.\begin{aligned} \mathbf{u'}(t+T) &= \mathbf{u'}(t), \\ \varphi(t+T) &= \varphi(t) + 2\pi\nu, \\ h(t+T) &= h(t), \quad h(0) = h(T) = 1, \end{aligned}\right\} \tag{3}$$

for all t (the smallest such number T is called the period of the motion), then the motion E/E' is called a *1-parameter closed planar homothetic motion*, where the integer ν is the number of rotations of the closed planar homothetic motion. If an equation in (3) is not satisfied, then the motion is called an open homothetic motion. During such a motion, the trajectories of the points are open curves. The orientated surface area swept out by a fixed line segment is studied by [8] and is generalized to the open homothetic motions by [7]. The polar moments of inertia of the open curves under the open homothetic motions can be given in a similar way.

Since $\dot{\varphi}(t) = 0$ gives pure translation, we assume that

$$\frac{d\varphi}{dt} = \dot{\varphi}(t) \neq 0$$

during the homothetic motion E/E'.

If we substitute Eq.(2) into Eq.(1) and then differentiate Eq.(1) with respect to t, we get the sliding velocity of a fixed point $X \in E$ as

$$V_{\mathbf{f}} = (\dot{h} + ih\dot{\varphi})\mathbf{x}e^{i\varphi} - (\dot{\mathbf{u}} + i\mathbf{u}\dot{\varphi})e^{i\varphi}.$$

Then, during E/E', using $V_{\mathbf{f}} = 0$, we get the moving pole point $\mathbf{p} = p_1 + ip_2$ as

$$\mathbf{p} = \frac{\dot{\mathbf{u}} + i\mathbf{u}\dot{\varphi}}{\dot{h} + ih\dot{\varphi}}$$

and we find for $\mathbf{u}$

$$u_1 = p_1 h + \frac{p_2 dh}{d\varphi} - \frac{du_2}{d\varphi} \quad , \quad u_2 = p_2 h - \frac{p_1 dh}{d\varphi} + \frac{du_1}{d\varphi}. \tag{4}$$

We assume that $\nu > 0$ throughout this study.

The Steiner point S, which is the center of gravity of the moving pole curve (P) for the distribution of mass with density $h^2 d\varphi$, is given by

$$\mathbf{s} = s_1 + is_2 = \frac{\oint \mathbf{p} h^2 d\varphi}{\oint h^2 d\varphi}, \tag{5}$$

where the integrations are taken along the closed pole curve (P).

Furthermore, using the mean-value theorem for integration of a continuous function, we have

$$\oint h^2 d\varphi = 2h_0^2 \pi \nu, \tag{6}$$

where $h_0 := h(t_0), \quad t_0 \in [0, T]$.

3 The Polar Moment of Inertia of the Orbit Curve

I.

Let X be a fixed point in E and (X) be the orbit curve of X. Then, the *polar moment of inertia* T_X of (X) is given by

$$T_X = \oint \mathbf{x}' \overline{\mathbf{x}'} d\varphi, \tag{7}$$

where $\mathbf{x}'$ is given by Eq.(1) and the integration is taken along the closed orbit curve (X) in E', [3].

Using the equations

$$\mathbf{u}' = -\mathbf{u}e^{i\varphi} \quad , \quad \overline{\mathbf{u}'} = -\overline{\mathbf{u}}e^{-i\varphi},$$

from Eq.(1) we get

$$\mathbf{x}'\overline{\mathbf{x}'} = \mathbf{u}'\overline{\mathbf{u}'} - h\mathbf{u}\overline{\mathbf{x}} - h\overline{\mathbf{u}}\mathbf{x} + h^2\mathbf{x}\overline{\mathbf{x}}. \tag{8}$$

Hence, by substituting Eq.(8) into Eq.(7), we obtain

$$T_X = \oint \mathbf{u}'\overline{\mathbf{u}'}d\varphi - 2x_1 \oint u_1 h d\varphi - 2x_2 \oint u_2 h d\varphi + \mathbf{x}\overline{\mathbf{x}} \oint h^2 d\varphi. \tag{9}$$

If $X = O(x_1 = x_2 = 0)$ is taken, then, for the polar moment of inertia of the origin point O we have

$$T_O = \oint \mathbf{u}'\overline{\mathbf{u}'}d\varphi. \tag{10}$$

Substituting the Eqs.(4), (5), (6) and (10) into Eq.(9) yields

$$T_X = T_O + 2h_0^2\pi\nu(\mathbf{x}\overline{\mathbf{x}} - \mathbf{x}\overline{\mathbf{s}} - \overline{\mathbf{x}}\mathbf{s}) + (\mathbf{x}\overline{\boldsymbol{\eta}} + \overline{\mathbf{x}}\boldsymbol{\eta}), \tag{11}$$

where $\overline{\mathbf{x}}$, $\overline{\mathbf{s}}$ and $\overline{\boldsymbol{\eta}}$ are complex conjugates of $\mathbf{x}$, $\mathbf{s}$ and $\boldsymbol{\eta}$, respectively, $\boldsymbol{\eta} = \eta_1 + i\eta_2$ and

$$\eta_1 = \oint(-hp_2dh + hdu_2) \quad , \quad \eta_2 = \oint(hp_1dh - hdu_1). \tag{12}$$

Then, we may give the following theorem.

THEOREM 1. Let us consider the 1-parameter closed planar homothetic motions. All the fixed points of the moving plane whose orbit curves have equal polar moment of inertia lie on the same circle with the center

$$\mathbf{c} = \mathbf{s} - \frac{1}{2h_0^2\pi\nu}\boldsymbol{\eta} \tag{13}$$

in the moving plane.

SPECIAL CASE 1. In the case of $h(t) \equiv 1$, we have $\boldsymbol{\eta} = 0$. Thus, we get

$$T_X = T_O + 2\pi\nu(\mathbf{x}\overline{\mathbf{x}} - \mathbf{x}\overline{\mathbf{s}} - \overline{\mathbf{x}}\mathbf{s}),$$

which was given by H. R. Müller. Also, the center C and the Steiner point S coincide, [3].

II.

Let X and Y be two fixed points in E, and Z be an arbitrary fixed point on the line segment XY, that is,

$$\mathbf{z} = \lambda\mathbf{x} + \xi\mathbf{y}, \qquad \lambda + \xi = 1.$$

Using Eq.(1) we have

$$\mathbf{z}' = \lambda\mathbf{x}' + \xi\mathbf{y}'.$$

Then, the polar moment of inertia T_Z of the curve (Z) is obtained as

$$T_Z = \oint \mathbf{z}'\overline{\mathbf{z}'}d\varphi = \lambda^2 T_X + 2\lambda\xi T_{XY} + \xi^2 T_Y, \tag{14}$$

where

$$T_{XY} = T_{YX} = \frac{1}{2}\oint (\mathbf{x}'\overline{\mathbf{y}'} + \overline{\mathbf{x}'}\mathbf{y}')d\varphi \tag{15}$$

is called the *mixture polar moment of inertia* of the curves (X) and (Y), [3].

If we use Eq.(1) in Eq.(15), the mixture polar moment of inertia is found as

$$T_{XY} = T_O + h_0^2\pi\nu[\mathbf{x}\overline{\mathbf{y}} + \overline{\mathbf{x}}\mathbf{y} - (\mathbf{x}+\mathbf{y})\overline{\mathbf{s}} - (\overline{\mathbf{x}}+\overline{\mathbf{y}})\mathbf{s}] + \frac{1}{2}[(\mathbf{x}+\mathbf{y})\overline{\boldsymbol{\eta}} + (\overline{\mathbf{x}}+\overline{\mathbf{y}})\boldsymbol{\eta}]. \tag{16}$$

It is clearly seen that $T_{XX} = T_X$.

SPECIAL CASE 2. In the case of $h(t) \equiv 1$, since $\boldsymbol{\eta} = 0$, we get

$$T_{XY} = T_O + \pi\nu[\mathbf{x}\overline{\mathbf{y}} - \overline{\mathbf{x}}\mathbf{y} - (\mathbf{x}+\mathbf{y})\overline{\mathbf{s}} - (\overline{\mathbf{x}}+\overline{\mathbf{y}})\mathbf{s}],$$

which was given by [3].

Let the origin point O and the point C given by Eq.(13) be coincident, i.e. $\mathbf{s} = \boldsymbol{\eta}/2h_0^2\pi\nu$. In this case, we get $T_O = T_C$ and from the Eqs.(11) and (16) we obtain

$$\left.\begin{array}{ll} T_X & = 2h_0^2\pi\nu\mathbf{x}\overline{\mathbf{x}} + T_C \\ T_{XY} & = h_0^2\pi\nu(\mathbf{x}\overline{\mathbf{y}} + \overline{\mathbf{x}}\mathbf{y}) + T_C \end{array}\right\}. \tag{17}$$

Thus, from latter equations, we get

$$T_X > T_C \quad \text{for} \quad X \neq C \tag{18}$$

and

$$T_X - 2T_{XY} + T_Y = 2h_0^2\pi\nu d_{XY}^2 \quad \text{for} \quad X \neq Y, \tag{19}$$

where d_{XY} is the distance between the points X and Y. By the orientation of the line XY we will distinguish $d_{XY} = -d_{YX}$.

COROLLARY 1. From Eq.(18), we can say that the orbit curve of the point C has the minimum polar moment of inertia.

From Eqs.(14) and (19), we get

$$T_Z = \lambda T_X + \xi T_Y - 2\lambda\xi h_0^2\pi\nu d_{XY}^2. \tag{20}$$

4 The Holditch Type Theorem for Polar Moments of Inertia

I.

THEOREM 2. Let us consider a line segment XY with constant length. If the endpoints X and Y trace the same closed convex curve in the fixed plane during the 1-parameter planar homothetic motion, then, the point Z on this line segment traces another closed curve. The difference between the polar moments of inertia of these curves depends on the distances of Z from the endpoints and the homothetic scale of the motion.

PROOF. Since X, Y and Z are collinear, we may write

$$d_{XZ} + d_{ZY} = d_{XY}.$$

Thus, if we denote

$$\lambda = \frac{d_{ZY}}{d_{XY}}, \quad \xi = \frac{d_{XZ}}{d_{XY}},$$

from Eq.(20) taking $\nu = 1$, we get

$$T_Z = \frac{1}{d_{XY}}\left(d_{ZY}T_X + d_{XZ}T_Y\right) - 2h_0^2\pi d_{XZ}d_{ZY}. \tag{21}$$

Since X and Y trace the same closed curve, we have $T_X = T_Y$. Then, from Eq.(21) we obtain

$$T_X - T_Z = 2h_0^2\pi d_{XZ}d_{ZY} \tag{22}$$

which is the equivalent formula to Holditch's result on areas[1], [2].

SPECIAL CASE 3. In the case of $h(t) \equiv 1$, we get the result given by [3].

II.

Let X_1, X_2 and X_3 be noncollinear fixed points in the moving plane E. Then, for any fixed point Q in E (Figure 2), we may write

$$\mathbf{q} = \lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \lambda_3\mathbf{x_3}, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{23}$$

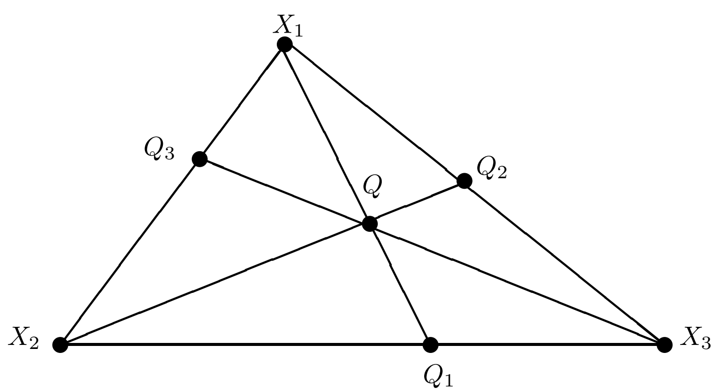

Figure 2. The fixed triangle in the moving plane

[1] The classical Holditch Theorem: If the endpoints X, Y of a segment of fixed length are rotated once on an oval, then a given point Z of this segment, with $\overline{XZ} = a$, $\overline{ZY} = b$, describes a closed, not necessarily convex, curve. The area of the ring-shaped domain bounded by the two curves is πab.

If we use Eq.(11), we obtain

$$T_Q = \lambda_1^2 T_{X_1} + \lambda_2^2 T_{X_2} + \lambda_3^2 T_{X_3} + 2\lambda_1\lambda_2 T_{X_1X_2} + 2\lambda_1\lambda_3 T_{X_1X_3} + 2\lambda_2\lambda_3 T_{X_2X_3}. \tag{24}$$

After eliminating the mixture polar moments of inertia by using Eq.(19), we get

$$T_Q = \lambda_1 T_{X_1} + \lambda_2 T_{X_2} + \lambda_3 T_{X_3} - 2h_0^2\pi\nu \left\{\lambda_1\lambda_2 d^2_{X_1X_2} + \lambda_1\lambda_3 d^2_{X_1X_3} + \lambda_2\lambda_3 d^2_{X_2X_3}\right\}. \tag{25}$$

On the other hand, if we consider the point Q_1, we may write

$$\mathbf{q_1} = \mu_1 \mathbf{x_2} + \mu_2 \mathbf{x_3}, \quad \mathbf{q} = \mu_3 \mathbf{x_1} + \mu_4 \mathbf{q_1}, \quad \mu_1 + \mu_2 = \mu_3 + \mu_4 = 1.$$

Thus, we have $\lambda_1 = \mu_3,\ \lambda_2 = \mu_1\mu_4,\ \lambda_3 = \mu_2\mu_4$, i.e.

$$\lambda_1 = \frac{d_{QQ_1}}{d_{X_1Q_1}}, \quad \lambda_2 = \frac{d_{X_1Q} d_{Q_1X_3}}{d_{X_1Q_1} d_{X_2X_3}}, \quad \lambda_3 = \frac{d_{X_1Q} d_{X_2Q_1}}{d_{X_1Q_1} d_{X_2X_3}}.$$

Similarly, considering the points Q_2 and Q_3, respectively, we find

$$\lambda_i = \frac{d_{QQ_i}}{d_{X_iQ_i}} = \frac{d_{X_jQ} d_{X_kQ_j}}{d_{X_jQ_j} d_{X_kX_i}} = \frac{d_{X_kQ} d_{Q_kX_j}}{d_{X_kQ_k} d_{X_iX_j}}, \quad i,j,k = 1,2,3(cyclic)$$

as given by [4]. Then, from Eq.(25) the generalization of Eq.(20) is found as

$$T_Q = \sum \frac{d_{QQ_i}}{d_{X_iQ_i}} T_{X_i} - 2h_0^2\pi\nu \sum \left(\frac{d_{X_kQ}}{d_{X_kQ_k}}\right)^2 d_{Q_kX_j} d_{X_iQ_k}. \tag{26}$$

This expression also generalizes the result given for the polar moments of inertia of the projection curves, [6].

If X_1, X_2, X_3 trace the same closed curve, then the difference between the polar moments of inertia is

$$T_{X_1} - T_Q = 2h_0^2\pi \sum \left(\frac{d_{X_kQ}}{d_{X_kQ_k}}\right)^2 d_{Q_kX_j} d_{X_iQ_k}.$$

THEOREM 3. Let us consider a fixed triangle in the moving plane. If the vertices of this triangle trace the same closed curve in the fixed plane during the 1-parameter planar homothetic motion, then, a different point in the moving plane traces another closed curve. The difference between the polar moments of inertia of these curves depends on the distances of the moving triangle and the homothetic scale.

EXAMPLE. Let us consider the 1-parameter planar homothetic motion with $\mathbf{u}(t) = sint$, $h(t) = cost$ and $\varphi(t) = t$. In this case, the motion has the period $T = 2\pi$ and the rotation number $\nu = 1$. Also, we find

$$\mathbf{p} = \mathbf{s} = -i, \quad \boldsymbol{\eta} = -\pi i, \quad T_O = \pi$$

and $h_0^2 = cos^2 t_0 = \frac{1}{2},\ t_0 = \frac{\pi}{4}, \frac{3\pi}{4} \in [0, 2\pi]$. Now, let us find the polar moments of inertia of the orbit curves of the points $\mathbf{x} = i$ and $\mathbf{x} = 2+i$. The point $\mathbf{x} = i$ draws the circle with radius 1 and center at O'; the point $\mathbf{x} = 2+i$ draws the circle with radius $\sqrt{2}$ and center at $\mathbf{1}$ during the homothetic motion (Figure 3). The polar moments of inertia of these curves are found from Eq.(11) as

$$T_i = 2\pi, \quad T_{2+i} = 6\pi.$$

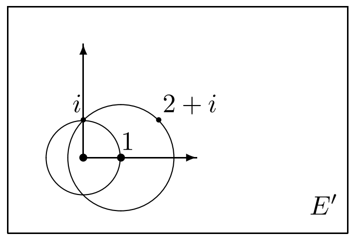

Figure 3. The orbit curves in the fixed plane

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References

[1] O. Bottema and B. Roth, Theoretical Kinematics, Amsterdam/Oxford/New York, 1979.

[2] H. Holditch, Geometrical Theorem, Q. J. Pure Appl. Math., 2(1858), 38.

[3] H. R. Müller, Über Trägheitsmomente bei Steinerscher Massenbelegung, Abh. Braunschw. Wiss. Ges., 29(1978), 115-119.

[4] H. R. Müller, Räumliche Gegenstücke zum Satz von Holditch, Abh. Braunschw. Wiss. Ges., 30(1979), 54-61.

[5] H. R. Müller, Erweiterung des Satzes von Holditch für geschlossene Raumkurven, Abh. Braunschw. Wiss. Ges., 31(1980), 129–135.

[6] M. Düldül and N. Kuruoğlu, On the polar moment of inertia of the projection curve, Appl. Math. E-Notes, 5(2005), 124-128.

[7] S. Yüce and N. Kuruoğlu, The Steiner formulas for the open planar homothetic motions, Appl. Math. E-Notes, 6(2006), 26-32.

[8] W. Blaschke and H. R. Müller, Ebene Kinematik, Oldenbourg, München, 1956.