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On Some Common Fixed Point Theorems Of Aamri And El Moutawakil In Uniform Spaces*

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Abstract

In this paper, we establish some common fixed point theorems for selfmappings in uniform spaces by employing the concepts of an A-distance and an E-distance introduced by Aamri and El Moutawakil [3] as well as the notion of comparison functions. We employ a more general contractive definition than that of Aamri and El Moutawakil [3]. Our results are generalizations of Theorems 3.1-3.3 of [3] and those of Olatinwo [12, 13].

1 Introduction

Let (X, Φ) be a uniform space, where X is a nonempty set equipped with a nonempty family Φ of subsets of $X \times X$ satisfying certain properties. Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.* Interested readers can consult Bourbaki [7] and Zeidler [20] for the definition of uniform space. The definition is also available on internet (by Wikipedia, the free encyclopedia).

The concept of a W-distance on metric space was introduced by Kada et al [10] to generalize some important results in nonconvex minimizations and in fixed point theory for both W-contractive and W-expansive maps. The theory of fixed point or common fixed point for contractive or expansive selfmappings in complete metric space has been well-developed. Interested readers can consult Berinde [5, 6], El Moutawakil [1], Aamri and El Moutawakil [2], Aamri et al [4], Jachymski [8], Kada et al [10], Kang [11], Rhoades [15], Rus [17], Rus et al [18], Wang et al [19] and Zeidler [20] for further study of fixed point or common fixed point theory.

Using the ideas of Kang [11], Montes and Charris [16] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space. Furthermore, Aamri and El Moutawakil [3] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

In [3], the following contractive definition was employed: Let $f,\ g : X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))),\ \ \forall x, y \in X, \tag{1}$$

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where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying (i) for each $t \in (0, +\infty)$, $0 < \psi(t)$, (ii) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \in (0, +\infty)$. ψ satisfies also the condition $\psi(t) < t$ for each $t > 0$.

In this paper, we shall establish some common fixed point theorems by employing a more general contractive condition than (1). We shall employ the concepts of A-distance and E-distance as well as the notion of comparison function in this work. Berinde [5, 6] extended the Banach's fixed point theorem using different contractive definitions involving the concept of the comparison functions. Rus [17] and Rus et al [18] also contain various generalizations and extensions of the Banach's fixed point theorem in which the contractive conditions involve some comparison functions.

We obtain more general results than those of Theorems 3.1-3.3 of [3] as well as Olatinwo [12, 13].

2 Preliminaries

We shall require the following definitions and lemma in the sequel. Let (X, Φ) be a uniform space.

REMARK 2.1. When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

DEFINITION 2.2. If $V \in \Phi$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V-close. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a Cauchy sequence for Φ if for any $V \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$.

DEFINITION 2.3. A function $p : X \times X \to \mathbb{R}^+$ is said to be an A-distance if for any $V \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

DEFINITION 2.4. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if

(p_1) p is an $A-$distance,

(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y \in X.$

DEFINITION 2.5. A uniform space (X, Φ) is said to be Hausdorff if and only if the intersection of all $V \in \Phi$ reduces to the diagonal $\{(x, x) | \; x \in X\}$, i.e. if $(x, y) \in V$ for all $V \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \Phi$ is said to be symmetrical if $V = V^{-1} = \{(y, x) | \; (x, y) \in V\}$.

DEFINITION 2.6. Let (X, Φ) be a uniform space and p be an A-distance on X.

(i) X is said to be S-complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(iv) $f : X \to X$ is $\tau(\Phi)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.

(v) X is said to be p-bounded if $\delta_p(X) = \sup\{p(x, y) | \; x, y \in X\} < \infty$.

DEFINITION 2.7. Let (X,Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be p-compatible if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim\limits_{n\to\infty} p(f(x_n),u) = \lim\limits_{n\to\infty} p(g(x_n),u) = 0$ for some $u \in X$, then we have $\lim\limits_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

We shall also state the following definition of a comparison function which is required in the sequel to establish some common fixed point results in uniform space.

DEFINITION 2.8. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if: (i) ψ is monotone increasing; (ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0,\ \forall t \geq 0$.

REMARK 2.9. Every comparison function satisfies the condition $\psi(0) = 0$.

Also, both conditions (i) and (ii) imply that $\psi(t) < t,\ \forall t > 0$.

For more on the comparison functions, see Berinde [5, 6], Rus [17] and Rus et al [18].

In this paper, we shall employ the following contractive definition: Let $f,\ g : X \to X$ be selfmappings of X. There exist a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\varphi(0) = 0$, and a comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq \varphi(p(x, g(x))) + \psi(p(g(x), g(y))). \tag{2}$$

REMARK 2.10. The contractive condition (2) is more general than (1) in the sense that if in (2), $\varphi(u) = 0,\ \forall u \in \mathbb{R}^+$, then we obtain (1) stated in this paper which was employed by Aamri and El Moutawakil [3].

LEMMA 2.11. Let (X,Φ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}_{n=0}^{\infty},\ \{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty},\ \{\beta_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n,\ \forall n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n,\ \forall n \in \mathbb{N}$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n\ \forall m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X,Φ).

REMARK 2.12. A sequence in X is p-Cauchy if it satisfies the usual metric condition.

3 Main Results

The main results of this paper are the following:

THEOREM 3.1. Let (X,Φ) be a Hausdorff uniform space and p an A-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_n = f(x_{n-1}),\ n = 1, 2, ...,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that (i) $f(X) \subseteq g(X)$; (ii) $p(f(x_i), f(x_i)) = 0,\ \forall x_i \in X,\ i = 0, 1, 2, ...$; and (iii) $f, g : X \to X$ satisfy the contractive condition (2). Suppose also that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. Then, f and g have a common fixed point.

PROOF. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$, choose $x_1 \in X$ such that $f(x_1) = g(x_2)$, and in general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. We recall that $x_n = f(x_{n-1}),\ n = 1, 2, ...$, so that by conditions (ii) and (iii) of the Theorem, we obtain

$$\begin{aligned}
p(f(x_n), f(x_{n+m})) &\leq \varphi(p(x_n, g(x_n))) + \psi(p(g(x_n), g(x_{n+m}))) \\
&= \varphi(p(f(x_{n-1}), f(x_{n-1}))) + \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&= \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&\leq \psi(\varphi(p(x_{n-1}, g(x_{n-1}))) + \psi(p(g(x_{n-1}), g(x_{n+m-1})))) \\
&= \psi(\varphi(p(f(x_{n-2}), f(x_{n-2}))) + \psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
&= \psi(\psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
&= \psi^2(p(f(x_{n-2}), f(x_{n+m-2})) \\
&\leq \cdots \leq \psi^n(p(f(x_0), f(x_m)) \leq \psi^n(\delta_p(X)),
\end{aligned}$$

from which we have that

$$p(f(x_n), f(x_{n+m})) \leq \psi^n(\delta_p(X)), \tag{3}$$

where $p(f(x_0), f(x_m)) \leq \delta_p(X)$ and $\delta_p(X) = \sup\{p(x,y)|\ x, y \in X\} < \infty$. Therefore, using the definition of comparison function in (3) yields $\psi^n(\delta_p(X)) \to 0$ as $n \to \infty$, from which it follows that $p(f(x_n), f(x_{n+m})) \to 0$ as $n \to \infty$. Hence, by applying Lemma 2.11(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence. Since X is S-complete, $\lim\limits_{n\to\infty} p(f(x_n), u)) = 0$, for some $u \in X$, and therefore $\lim\limits_{n\to\infty} p(g(x_n), u)) = 0$. Since f and g are p-continuous, then

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0. \tag{4}$$

Also, since f and g are commuting, that is, $fg = gf$, then we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0, \tag{5}$$

so that by Lemma 2.11(a), we obtain that $f(u) = g(u)$. Since $f(u) = g(u),\ fg = gf$, we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. Using (2) and the condition $\psi(t) < t,\ \forall t > 0$ in the Remark 2.9, then, we have

$$\begin{aligned}
p(f(u), f(f(u))) &\leq \varphi(p(u, g(u))) + \psi(p(g(u), g(f(u)))) \\
&= \varphi(p(f(u), f(u))) + \psi(p(f(u), f(f(u)))) \\
&= \psi(p(f(u), f(f(u))) < p(f(u), f(f(u))),
\end{aligned}$$

which is a contradiction. Therefore, $p(f(u), f(f(u))) = 0$. Condition (ii) of the Theorem yields $p(f(u), f(u)) = 0$. Since $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, applying Lemma 2.11(a) then yields $f(f(u)) = f(u)$. Thus, we have $g(f(u)) = f(f(u)) = f(u)$. Hence, $f(u)$ is a common fixed point of f and g. The proof is similar when f and g are $\tau(\Phi)$-continuous as S-completeness implies p-Cauchy completeness.

REMARK 3.2. Theorem 3.1 is a generalization of Theorem 3.1 of Aamri and El Moutawakil [3] and similar results in Olatinwo [12, 13].

THEOREM 3.3. Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_n = f(x_{n-1}),\ n = 1, 2, ...,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that (i) $f(X) \subseteq g(X)$; (ii) $p(f(x_i), f(x_i)) = 0,\ \forall x_i \in X,\ i = 0, 1, 2, ...,$ and (iii) $f,\ g : X \to X$ satisfy the contractive condition (2). Suppose also that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. Then, f and g have a unique common fixed point.

PROOF. f and g have a common fixed point since an E-distance function p is an A-distance. Suppose that there exist $u,\ v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. Let $p(u, v) \neq 0$. Then, we have

$$\begin{aligned} p(u, v) &= p(f(u), f(v)) \leq \varphi(p(u, g(u))) + \psi(p(g(u), g(v))) \\ &= \varphi(p(u, u)) + \psi(p(u, v)) = \psi(p(u, v)) \\ &< p(u, v), \end{aligned}$$

which is a contradiction. Therefore, we have $p(u, v) = 0$. By carrying out a similar process, we also have that $p(v, u) = 0$. Using condition (p_2) of Definition 2.4, we have $p(u, u) \leq p(u, v) + p(v, u)$, from which it follows that $p(u, u) = 0$. Since $p(u, u) = 0$ and $p(u, v) = 0$, then by Lemma 2.11(a), we have that $u = v$.

REMARK 3.4. Theorem 3.3 is a generalization of Theorem 3.2 of Aamri and El Moutawakil [3] and similar ones in Olatinwo [12, 13].

THEOREM 3.5. Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_n = f(x_{n-1}),\ n = 1, 2, ...,$$

with $x_0 \in X$. Let f and g be p-compatible, p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that (i) $f(X) \subseteq g(X)$; (ii) $p(f(x_i), f(x_i)) = 0,\ \forall x_i \in X,\ i = 0, 1, 2, ...;$ and (iii) $f,\ g : X \to X$ satisfy the contractive condition (2). Suppose also that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. Then, f and g have a unique common fixed point.

PROOF. Just as in the proof of Theorem 3.1, we have for some $u \in X$,

$$\lim_{n\to\infty} p(f(x_n, u)) = \lim_{n\to\infty} p(g(x_n, u)) = 0.$$

Since f and g are p-continuous, we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0,$$

while the assumption that f and g are p-compatible implies $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$. Furthermore, by condition (p_2) of Definition 2.4, we have that

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)). \tag{6}$$

Taking limits in (6) and applying Lemma 2.11(a), then we have $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$. Since $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 2.11(a) we have $f(u) = g(u)$. The rest of the proof is as in Theorem 3.3.

REMARK. 3.6. Theorem 3.5 is also a generalization of Theorem 3.3 of Aamri and El Moutawakil [3] and similar ones in Olatinwo [12, 13]. We refer our interested readers to Olatinwo [14] for common fixed point theorems in uniform space involving contractive conditions of integral type.

EXAMPLE 3.7. Let $X = [0,1]$ and $d(x,y) = |x-y|,\ \forall x,y \in X$ (where d is the usual metric on $\mathbb{R}$). Let f and g be defined by

$$f(x) = \begin{cases} \frac{1}{2} - x, & x \in [0, \frac{1}{2}) \\ 0, & x \in [\frac{1}{2}, 1] \end{cases},$$

and

$$g(x) = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ \frac{3}{4}, & x \in [\frac{1}{2}, 1] \end{cases}.$$

Suppose also that p, ψ and φ are respectively given by

$$p(x,y) = \begin{cases} y^2, & y \in [0, \frac{1}{2}) \\ 1, & y \in [\frac{1}{2}, 1] \end{cases},$$

$$\psi(x) = \begin{cases} \frac{1}{5}x, & x \in [0, \frac{1}{2}) \\ \frac{1}{4}x, & x \in [\frac{1}{2}, 1] \end{cases},$$

and

$$\varphi(x) = \begin{cases} \frac{1}{3}x, & x \in [0, \frac{1}{2}) \\ \frac{1}{4}, & x \in [\frac{1}{2}, 1] \end{cases}.$$

The function p is an E-distance, ψ is a comparison function, φ is a monotone increasing function and X is S-complete. In addition, f and g are commuting, p-continuous and that

$$\varphi\left(d\left(\frac{1}{4}, g\left(\frac{1}{4}\right)\right)\right) = \varphi(0) = 0$$

and

$$d\left(f\left(\frac{1}{4}\right), f\left(\frac{1}{5}\right)\right) = \frac{1}{20} > \psi\left(d\left(g\left(\frac{1}{4}\right), g\left(\frac{1}{5}\right)\right)\right) = \psi\left(\frac{1}{20}\right) = \frac{1}{100},$$

which implies that $p(f(x), f(y)) \le \varphi(p(x, g(x))) + \psi(p(g(x), g(y)))$ does not hold $\forall x, y \in X$.

On the other hand, we have that

$$p(f(x), f(y)) \le \varphi(p(x, g(x))) + \psi(p(g(x), g(y))),\ \forall x, y \in X$$

and $\frac{1}{4}$ is the unique common fixed point of f and g.

EXAMPLE 3.8. Let $X = [0,1]\times[0,1]$ and $\delta(x,y) = \sum_{j=1}^{2}|x_j - y_j|,\ \forall \vec{x},\vec{y}\in \mathbb{R}^2$ (where δ is a metric on $\mathbb{R}^2$). See Berinde [6] for the metric δ on $\mathbb{R}^{\mathrm{n}}$. Let f and g be defined by

$$f(x_1,x_2) = \begin{cases} (-x_1, \frac{1}{3}-x_2), & x_1,x_2 \in [0,\frac{1}{3})\times[0,\frac{1}{3}) \\ (0,0), & x_1,x_2\in[\frac{1}{3},1]\times[\frac{1}{3},1] \end{cases}$$

and

$$g(x_1,x_2) = \begin{cases} (x_1,x_2), & x_1,x_2 \in [0,\frac{1}{3})\times[0,\frac{1}{3}) \\ (\frac{3}{4},\frac{2}{5}), & x_1,x_2\in[\frac{1}{3},1]\times[\frac{1}{3},1] \end{cases}.$$

Suppose also that $p,\ \psi$ and φ are respectively given by

$$p(x,y) = \begin{cases} y^k, & y\in[0,\frac{1}{3}), k\geq 1 \\ 1, & y\in[\frac{1}{3},1] \end{cases},$$

$$\psi(x) = \begin{cases} \frac{1}{10}x, & x\in[0,\frac{1}{3}) \\ \frac{1}{5}x, & x\in[\frac{1}{3},1] \end{cases}$$

and

$$\varphi(x) = \begin{cases} \frac{1}{4}x, & x\in[0,\frac{1}{3}) \\ \frac{3}{5}x, & x\in[\frac{1}{3},1] \end{cases}$$

The function p is again an E-distance, ψ is a comparison function, φ is a monotone increasing function and X is S-complete. If we put $\vec{x} = (x_1,x_2)$ and re-write f and g as

$$f(\vec{x}) = \begin{cases} (0,\frac{1}{3})-\vec{x}, & x_1,x_2 \in [0,\frac{1}{3})\times[0,\frac{1}{3}) \\ (0,0), & x_1,x_2\in[\frac{1}{3},1]\times[\frac{1}{3},1] \end{cases}$$

and

$$g(\vec{x}) = \begin{cases} \vec{x}, & x_1,x_2 \in [0,\frac{1}{3})\times[0,\frac{1}{3}) \\ (\frac{3}{4},\frac{2}{5}), & x_1,x_2\in[\frac{1}{3},1]\times[\frac{1}{3},1] \end{cases},$$

then we see that f and g are commuting and p-continuous. Also, $g\left(0,\frac{1}{6}\right) = f\left(0,\frac{1}{6}\right) = \left(0,\frac{1}{6}\right)$, $\varphi\left(\delta\left(\left(0,\frac{1}{6}\right),\left(0,\frac{1}{6}\right)\right)\right) = 0$, $\delta(\vec{x},g(\vec{x})) = \delta\left(\left(0,\frac{1}{6}\right), g\left(0,\frac{1}{6}\right)\right) = \delta\left(\left(0,\frac{1}{6}\right),\left(0,\frac{1}{6}\right)\right) = 0$, and $\delta(f(\vec{x}),f(\vec{y})) = \delta\left(f\left(0,\frac{1}{6}\right), f\left(0,\frac{1}{5}\right)\right) = \delta\left(\left(0,\frac{1}{6}\right),\left(0,\frac{2}{15}\right)\right) = \frac{1}{30}$,

$$\begin{aligned} \psi(\delta(g(\vec{x}),g(\vec{y}))) &= \psi\left(\delta\left(g\left(0,\frac{1}{6}\right), g\left(0,\frac{1}{5}\right)\right)\right) \\ &= \psi\left(\delta\left(\left(0,\frac{1}{6}\right),\left(0,\frac{1}{5}\right)\right)\right) \\ &= \psi\left(\frac{1}{30}\right) \\ &= \frac{1}{300}. \end{aligned}$$

So, $\delta(f(\vec{x}),f(\vec{y})) = \frac{1}{30} > \varphi(0) + \psi(\delta(g(\vec{x}),g(\vec{y}))) = \frac{1}{300}$, which implies that

$$p(f(\vec{x}),f(\vec{y})) \leq \varphi(p(\vec{x},g(\vec{x}))) + \psi(p(g(\vec{x}),g(\vec{y})))$$

does not hold $\forall \vec{x}, \vec{y} \in X$. On the other hand, we have that

$$p(f(\vec{x}), f(\vec{y})) \leq \varphi(p(\vec{x}, g(\vec{x}))) + \psi(p(g(\vec{x}), g(\vec{y}))), \forall \vec{x}, \vec{y} \in X$$

and the vector $(0, \frac{1}{6})$ is the unique common fixed point of f and g, that is, $F_f \bigcap F_g = \left\{(0, \frac{1}{6})\right\}$, where F_f and F_g are the fixed point sets of f and g respectively.

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