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A Generalized Ostrowski Type Inequality For A Random Variable Whose Probability Density Function Belongs To $L_p[a,b]$*

Arif Rafiq†, Nazir Ahmad Mir‡, Fiza Zafar§

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Abstract

We establish here an inequality of Ostrowski type for a random variable whose probability density function belongs to $L_p[a,b]$, in terms of the cumulative distribution function and expectation. The inequality is then applied to generalized beta random variable.

1 Introduction

The following theorem describes an inequality which is known in literature as Ostrowski inequality [7].

THEOREM 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in I^0 (interior of I), and let $a, b \in I^0$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\left\|f'\right\|_\infty := \sup_{t\in(a,b)} \left|f'(t)\right| < \infty$, then we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\|f'\right\|_\infty, \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [2], N. S. Barnett and S. S. Dragomir established the following version of Ostrowski type inequality for cumulative and probability density functions.

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†Department of Mathematics, COMSATS Institute of Information Technology, Plot # 30, Sector H-8/1, Islamabad 44000, Pakistan.

‡Department of Mathematics, COMSATS Institute of Information Technology, Plot # 30, Sector H-8/1, Islamabad 44000, Pakistan.

§Centre for Advanced Studies in Pure and Applied Mathematics, B. Z. University, Multan 60800, Pakistan.

THEOREM 2. Let X be a random variable with probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}^+$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$. If $f \in L_\infty [a,b]$ and $\|f\|_\infty := \sup_{t \in [a,b]} |f(t)| < \infty$, then we have the inequality:

$$\left| \Pr(X \leq x) - \frac{b - E(X)}{b-a} \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f\|_\infty , \tag{2}$$

for all $x \in [a,b]$.

Equivalently,

$$\left| \Pr(X \geq x) - \frac{E(X) - a}{b-a} \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f\|_\infty . \tag{3}$$

The constant $\frac{1}{4}$ in (2) and (3) is sharp.

In [4], S. S. Dragomir, N. S. Barnett and S. Wang developed Ostrowski type inequality for a random variable whose probability density function belongs to $L_p[a,b]$ in terms of the cumulative distribution function and expectation. The inequality is given in the form of the following theorem:

THEOREM 3. Let X be a random variable with the probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}^+$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$. If $f \in L_p [a,b]$, $p > 1$, then we have the inequality:

$$\begin{aligned} \left| \Pr(X \leq x) - \frac{b - E(X)}{b-a} \right| &\leq \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}} \left[\left(\frac{x-a}{b-a} \right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a} \right)^{\frac{1+q}{q}} \right] \\ &\leq \frac{q}{1+q} \|f\|_p (b-a)^{\frac{1}{q}} , \end{aligned} \tag{4}$$

for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

In [8], we may find the following theorem:

THEOREM 4. Let $f : [a,b] \to \mathbb{R}$ be continuous, differentiable on $[a,b]$ and $f' \in L_p [a,b]$ for some $p > 1$. Then

$$\begin{aligned} & \left| (b-a) \left[(1-h) f(x) + h \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t)\, dt \right| \\ \leq & \frac{1}{(q+1)^{\frac{1}{q}}} \left[2 \left(\frac{h(b-a)}{2} \right)^{q+1} + \left(x - a - \frac{h(b-a)}{2} \right)^{q+1} \right. \\ & \left. + \left(b - x - \frac{h(b-a)}{2} \right)^{q+1} \right]^{\frac{1}{q}} \left\| f' \right\|_p , \end{aligned} \tag{5}$$

where $q = \frac{p}{p-1}$, $h \in [0,1]$ and $a + h\frac{b-a}{2} \leq x \leq b - h\frac{b-a}{2}$.

The main aim of this paper is to develop an Ostrowski type inequality for random variables whose probability density functions are in $L_p [a,b]$ based on (5). Applications for a generalized beta random variable are also given.

2 Main Results

The following theorem holds.

THEOREM 5. Let X and F be as defined above. Then from Theorem 4 we have

$$\begin{aligned}
&\left|(1-h)F(x)+\frac{h}{2}-\frac{1}{b-a}\int_a^b F(t)\,dt\right| \\
\leq\ & \frac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[2\left(\frac{h(b-a)}{2}\right)^{q+1}+\left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\left.+\left(b-x-\frac{h(b-a)}{2}\right)^{q+1}\right]^{\frac{1}{q}}\|f\|_p, \qquad (6)
\end{aligned}$$

where f is the probability density function associated with the cumulative distribution function F.

Equivalently,

$$\begin{aligned}
&\left|(1-h)\Pr(X\leq x)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right| \\
\leq\ & \frac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[2\left(\frac{h(b-a)}{2}\right)^{q+1}+\left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\left.+\left(b-x-\frac{h(b-a)}{2}\right)^{q+1}\right]^{\frac{1}{q}}\|f\|_p, \qquad (7)
\end{aligned}$$

for all $x\in[a+h\frac{b-a}{2},b-h\frac{b-a}{2}]$ and $h\in[0,1]$.

The proof is obvious. Hence, the details are omitted.

We now give some corollaries of the above theorem for the expectations of the variable X.

COROLLARY 1. Under the above assumptions, we have the double inequality

$$\begin{aligned}
& b-\frac{h}{2}(b-a)-\frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p \\
\leq\ & E(X) \\
\leq\ & a+\frac{h}{2}(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p, \qquad (8)
\end{aligned}$$

for $h\in[0,1]$ and

$$\triangle(q,h)=\left(\left(\frac{h}{2}\right)^{q+1}(2-(-1)^q)+\left(1-\frac{h}{2}\right)^{q+1}\right)^{\frac{1}{q}}. \qquad (9)$$

PROOF. It is known that $a \leq E(X) \leq b$. If $x = a$ in (7), we obtain

$$\left|\frac{h}{2} - \frac{b - E(X)}{b-a}\right| \leq \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \triangle(q,h)\|f\|_p,$$

implies

$$\begin{aligned} & b - \frac{h}{2}(b-a) - \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p \\ \leq \quad & E(X) \leq b - \frac{h}{2}(b-a) + \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p. \end{aligned} \tag{10}$$

The left hand estimate of the inequality (10) is equivalent to first inequality in (8).

Also, if $x = b$ in (7), then

$$\left|\frac{E(X) - a}{b-a} - \frac{h}{2}\right| \leq \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \triangle(q,h)\|f\|_p,$$

which reduces to

$$\begin{aligned} & a + \frac{h}{2}(b-a) - \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p \\ \leq \quad & E(X) \leq a + \frac{h}{2}(b-a) + \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p. \end{aligned} \tag{11}$$

The right hand side of the inequality (11) proves the second inequality of (8).

REMARK 1. As for the probability density function f associated with the random variable X,

$$1 = \int_a^b f(t)\,dt,$$

implies

$$\|f\|_p \geq \frac{1}{(b-a)^{\frac{1}{q}}}.$$

If we suppose that f is not 'too large' so that

$$\|f\|_p \leq \frac{(q+1)^{\frac{1}{q}}\left(1 - \frac{h}{2}\right)}{(b-a)^{\frac{1}{q}}\triangle(q,h)}, \tag{12}$$

then from the double inequality (8), it can be verified that

$$a + \frac{h}{2}(b-a) + \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p \leq b,$$

and

$$b - \frac{h}{2}(b-a) - \frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p \geq a,$$

when (12) holds. It shows that (8) gives a much tighter estimate of the expected value of the random variable X.

COROLLARY 2. With the above assumptions,

$$\left|E\left(X\right)-\frac{a+b}{2}\right|\leq\left(b-a\right)\left[\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\triangle\left(q,h\right)\left\|f\right\|_{p}-\frac{1-h}{2}\right],\tag{13}$$

where $\triangle\left(q,h\right)$ is defined by (9).

PROOF. From the inequality (8),

$$\begin{aligned}&\frac{1}{2}\left(b-a\right)\left(1-h\right)-\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\triangle\left(q,h\right)\left(b-a\right)^{1+\frac{1}{q}}\left\|f\right\|_{p}\\ \leq\ & E\left(X\right)-\frac{a+b}{2}\\ \leq\ & -\frac{1}{2}\left(b-a\right)\left(1-h\right)+\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\triangle\left(q,h\right)\left(b-a\right)^{1+\frac{1}{q}}\left\|f\right\|_{p}\end{aligned}$$

which is exactly (13).

This corollary provides the mechanism for finding a sufficient condition, in terms of $\left\|f\right\|_{p}$, for the expectation $E\left(X\right)$ to be close to the midpoint of the interval, $\frac{a+b}{2}$.

COROLLARY 3. Let X and f be as above and $\varepsilon>0$. If

$$\left\|f\right\|_{p}\leq\frac{\left(1-h\right)\left(q+1\right)^{\frac{1}{q}}}{2\triangle\left(q,h\right)\left(b-a\right)^{\frac{1}{q}}}+\frac{\left(q+1\right)^{\frac{1}{q}}\varepsilon}{\triangle\left(q,h\right)\left(b-a\right)^{1+\frac{1}{q}}},\tag{14}$$

then

$$\left|E\left(X\right)-\frac{a+b}{2}\right|\leq\varepsilon$$

The following corollary of Theorem 5 also holds.

COROLLARY 4. Let X and F be as above, then

$$\begin{aligned}&\left|\left(1-h\right)\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\left(1-h\right)\right|\\ \leq\ &\frac{1}{2}\left(h^{q+1}+\left(1-h\right)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\left\|f\right\|_{p}\\ &+\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\triangle\left(q,h\right)\left\|f\right\|_{p}-\frac{1}{2}\left(1-h\right).\end{aligned}\tag{15}$$

PROOF. If we choose $x=\frac{a+b}{2}$ in (7), then we get

$$\begin{aligned}&\left|\left(1-h\right)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E\left(X\right)}{b-a}\right|\\ \leq\ &\frac{1}{2}\left(h^{q+1}+\left(1-h\right)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\left\|f\right\|_{p},\end{aligned}$$

which may be rewritten in the following form

$$\begin{aligned}&\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{1}{2}\left(h^{q+1}+(1-h)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f\|_{p}.\end{aligned}$$

Using the triangular inequality, we get

$$\begin{aligned}&\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|\\ \leq\ &\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|\\ &+\frac{1}{b-a}\left|E\left(X\right)-\frac{a+b}{2}\right|\end{aligned}$$

gives the desired result.

A similar inequality holds for $\Pr\left(X\geq\frac{a+b}{2}\right)$.

3 Applications for Generalized Beta Random Variable

If X is a beta random variable with parameters $\beta_3>-1,\ \beta_4>-1$ and for $\beta_2>0$ and any β_1, the generalized beta random variable $Y=\beta_1+\beta_2X$, is said to have a generalized beta distribution [6] and the probability density function of the generalized beta distribution of beta random variable is,

$$f\left(x\right)=\begin{cases}\frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{B(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}} & \text{for } \beta_1<x<\beta_1+\beta_2\\ 0 & \text{otherwise.}\end{cases},$$

where $B\left(l,m\right)$ is the beta function with $l,m>0$ and is defined as

$$B\left(l,m\right)=\int_0^1 x^{l-1}\left(1-x\right)^{m-1}dx.$$

For $s,t>0$ and $h\in[0,1)$, we choose

$$\beta_1=\frac{h}{2},\ \beta_2=(1-h),\ \beta_3=s-1,\ \beta_4=t-1.$$

Then, the probability density function associated with generalized beta random variable $Y=\frac{h}{2}+(1-h)X$, takes the form

$$f\left(x\right)=\begin{cases}\frac{\left(x-\frac{h}{2}\right)^{s-1}\left(1-\frac{h}{2}-x\right)^{t-1}}{B(s,t)(1-h)^{s+t-1}} & \frac{h}{2}<x<1-\frac{h}{2}\\ 0 & \text{otherwise.}\end{cases}.$$

Now,

$$E(Y)=\int_{\frac{h}{2}}^{1-\frac{h}{2}} xf(x)\,dx=(1-h)\frac{s}{s+t}+\frac{h}{2}, \tag{16}$$

and

$$\|f\|_p=\frac{1}{(1-h)^{1-\frac{1}{p}}B(s,t)}B^{\frac{1}{p}}(p(s-1)+1,p(t-1)+1), \tag{17}$$

provided

$$s>1-\frac{1}{p},t>1-\frac{1}{p},$$

for $p>1$. Then, by Theorem 5, we may state the following.

PROPOSITION 1. Let X be a beta random variable with parameters (s,t). Then for generalized beta random variable $Y=\frac{h}{2}+(1-h)X$, we have the inequality

$$\begin{aligned}&\left|\Pr(Y\le x)-\frac{t}{s+t}\right|\\ \le\ & \frac{1}{(1-h)^{2-\frac{1}{p}}B(s,t)}\left(\frac{2\left(\frac{h}{2}\right)^{q+1}+\left(x-\frac{h}{2}\right)^{q+1}+\left(1-x-\frac{h}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}\times\\ & B^{\frac{1}{p}}(p(s-1)+1,p(t-1)+1),\end{aligned} \tag{18}$$

for all $x\in\left[\frac{h}{2},1-\frac{h}{2}\right]$.

In particular,

$$\begin{aligned}\left|\Pr\left(Y\le\frac{1}{2}\right)-\frac{t}{s+t}\right| \le\ & \frac{1}{2(1-h)^{2-\frac{1}{p}}B(s,t)}\left(\frac{h^{q+1}+(1-h)^{q+1}}{q+1}\right)^{\frac{1}{q}}\\ & \times B^{\frac{1}{p}}(p(s-1)+1,p(t-1)+1).\end{aligned} \tag{19}$$

REMARK 2. For $h=0$ in (18), we have the inequality

$$\begin{aligned}&\left|\Pr(X\le x)-\frac{t}{s+t}\right|\\ \le\ & \left(\frac{x^{q+1}+(1-x)^{q+1}}{q+1}\right)^{\frac{1}{q}}\frac{B^{\frac{1}{p}}(p(s-1)+1,p(t-1)+1)}{B(s,t)},\end{aligned} \tag{20}$$

for all $x\in[0,1]$, and particularly,

$$\left|\Pr\left(X\le\frac{1}{2}\right)-\frac{t}{s+t}\right|\le\frac{1}{2(q+1)^{\frac{1}{q}}}\frac{B^{\frac{1}{p}}(p(s-1)+1,p(t-1)+1)}{B(s,t)}.$$

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