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A Remark On A Parabolic Problem In A Sectorial Domain*

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Abstract

This work is concerned with the problem

$$\partial_t u - c^2(t)\partial_x^2 u = f, \quad u_{|\partial\Omega\setminus\Gamma_T} = 0$$

posed in the domain $\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\right\}$,which is not necessarily rectangular, and with $\Gamma_T = \{(T,x) : \varphi_1(T) < x < \varphi_2(T)\}$. Our aim is to present a new approach to find some conditions on the coefficient c and the functions $(\varphi_i)_{i=1,2}$ such that the solution of this problem belongs to the Sobolev space

$$H^{1,2}(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u \in L^2(\Omega),\ \partial_x u \in L^2(\Omega),\ \partial_x^2 u \in L^2(\Omega)\right\}.$$

The method makes use of the so-called Schur's Lemma and gives the same result proved in Sadallah [8] by another technique.

1 Introduction

In the domain $\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\right\}$, consider the problem

$$(P) \left\{ \begin{array}{c} \partial_t u - c^2(t)\partial_x^2 u = f \\ u_{|\partial\Omega\setminus\Gamma_T} = 0 \end{array} \right.$$

where

(i) $\Gamma_T = \{(T,x) : \varphi_1(T) < x < \varphi_2(T)\}$ if $T < +\infty$ and $\Gamma_T = \varnothing$ if $T = +\infty$,

(ii) c is a coefficient depending on time such that $0 < \alpha \leq c \leq \beta$, where α and β are two constants,

(iii) $(\varphi_i)_{i=1,2}$ are functions defined on $]0, T[$ and satisfy some assumptions to be made more precise later on,

(iv) $f \in L^2(\Omega)$ (usual Lebesgue space).

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We look for a solution u of Problem (P) in the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u,\ \partial_x u,\ \partial_x^2 u \in L^2(\Omega)\right\}.$$

The study of this kind of problems has been treated in [2-4,6,7,8]. The present work is concerned with the singular case where $\varphi_1(0) = \varphi_2(0)$. Observe that the techniques used, for instance, in [5] do not apply here because the domain is not cylindrical. This explains why the change of variables which we will perform leads to some degenerate coefficients in the equations.

The following is the main result proved in [8].

THEOREM 1. Suppose that the following conditions are satisfied ($T = +\infty$):

(a) $(\varphi_i)_{i=1,2}$ and c are continuous functions on $[0, +\infty[$, differentiable on $]\,0, +\infty[$ and $\varphi_1(0) = \varphi_2(0)$.

(b) $|\varphi_i'|\,(\varphi_2 - \varphi_1)$ is small enough in a neighborhood of 0 for $i = 1, 2$.

(c) $(\varphi_i')_{i=1,2}$ and c are bounded in a neighborhood of $+\infty$.

(d) $\varphi_2 - \varphi_1$ is increasing in a neighborhood of $+\infty$ or

$$\exists M > 0, |\varphi_1'(t) - \varphi_2'(t)|\,(\varphi_2(t) - \varphi_1(t)) \leq Mc(t).$$

Then Problem (P) admits a (unique) solution $u \in H^{1,2}(\Omega)$.

To prove this theorem, we have used some *a priori* estimates and divided the proof in four steps:

1) case of a bounded domain which can be transformed into a rectangle,
2) case of a bounded triangular domain,
3) case of an unbounded domain which can be transformed into a half strip,
4) case of a sectorial domain.

The aim of this work is to prove that we can combine the first two cases and study them using a new approach based on the so-called Schur's Lemma (see, for example [1]). This method consists in performing a change of variables conserving the spaces $L^2(\Omega)$ and $H^{1,2}(\Omega)$, and transforming Problem (P) into a degenerate parabolic problem in a cylindrical domain, and then conclude using Schur's Lemma.

2 Change of Variables

Assume that $\varphi_1(0) = \varphi_2(0)$ and $T < +\infty$. The change of variables

$$\begin{array}{ccc} \Omega & \to & R \\ (t,x) & \longmapsto & \left(t, \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right) = (t,y) \end{array}$$

transforms Ω into $R = \,]0,T[\,\times\,]0,1[$ and Problem (P) becomes

$$(P') \begin{cases} \partial_t v + a(t,y)\partial_y v - \frac{1}{b^2(t)}\partial_y^2 v = \widetilde{f} \\ v_{|\partial R \setminus \Gamma_T} = 0 \end{cases}$$

where

$$\begin{gathered}\widetilde{f}(t,y)=f(t,x),\\ a(t,y)=-\frac{y\left(\varphi_2'(t)-\varphi_1'(t)\right)+\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)},\\ b(t)=\frac{\varphi_2(t)-\varphi_1(t)}{c(t)}\geq 0.\end{gathered}$$

Observe that Problem (P) is equivalent to

$$(P_1)\ \left\{\begin{array}{c} b^2(t).\partial_t v-\frac{\varphi(t)\left(y\varphi'(t)+\varphi_1'(t)\right)}{c^2(t)}\partial_y v-\partial_y^2 v=h\\ v_{|\partial R\backslash\Gamma_T}=0\end{array}\right.$$

where $b^2(t).\widetilde{f}=h$ and $\varphi=\varphi_2-\varphi_1$. Let $H(R)$ and $H_{1,2}(R)$ be the spaces defined by

$$H(R)=\left\{f\in L^2(R):\frac{f}{\varphi^{\frac{3}{2}}}\in L^2(R)\right\},$$

$$H_{1,2}(R)=\left\{v\in H(R):\partial_y v,\ \partial_y^2 v,\ \varphi\partial_t v\in H(R), v_{|\partial R\backslash\Gamma_T}=0\right\}.$$

Then, consider the degenerate problem

$$(P_2)\ \left\{\begin{array}{c} b^2(t).\partial_t v-\partial_y^2 v=h\in H(R)\\ v_{|\partial R\backslash\Gamma_T}=0\end{array}\right. .$$

3 Degenerate Problem

We have the following result.

PROPOSITION 2. Assume that the function bb' is bounded and $\frac{2}{3}\pi^2>\sup|bb'|$. Then for all $h\in H(R)$, (P_2) admits a (unique) solution $v\in H_{1,2}(R)$.

PROOF. It is easy to check the uniqueness of the solution. Let us prove the existence. It is well known that the sequence (ψ_n) defined by $\psi_n(x)=\sqrt{2}\sin n\pi x$ in the interval $]0,1[$ is an orthonormal basis in $L^2(0,1)$ formed by eigenfunctions of the operator $-\partial_x^2$. Denote by $\lambda_n=n^2\pi^2$ the eigenvalue corresponding to ψ_n. Let

$$\begin{aligned} h(t,x) &= \sum_{n\geqslant 0} h_n(t)\psi_n(x),\\ v(t,x) &= \sum_{n\geqslant 0} v_n(t)\psi_n(x),\\ f_n &= \frac{h_n}{b^{\frac{3}{2}}}.\end{aligned}$$

The function v is a solution of (P_1) if

$$\forall n\in\mathbb{N},\quad b^2(t)v_n'+\lambda_n v_n=h_n.$$

So

$$v_n(t)=\int_0^t h_n(s)b^{-2}(s).e^{-\lambda_n\int_s^t\frac{dr}{b^2(r)}}ds. \tag{1}$$

Observe that $h \in H(R)$ if and only if

$$\sum_{n \geqslant 0} \|f_n\|^2_{L^2(0,T)} < +\infty, \tag{2}$$

and $v \in H_{1,2}(R)$ is equivalent to

$$\begin{aligned} &\sum_{n \geqslant 0} \left\| \sqrt{b} v_n \right\|^2_{L^2(0,T)} + \sum_{n \geqslant 0} \left\| \frac{\sqrt{\lambda_n}}{b^{\frac{3}{2}}} v_n \right\|^2_{L^2(0,T)} \\ &+ \sum_{n \geqslant 0} \left\| \frac{\lambda_n}{b^{\frac{3}{2}}} v_n \right\|^2_{L^2(0,T)} + \sum_{n \geqslant 0} \left\| b^{\frac{1}{2}} v_n' \right\|^2_{L^2(0,T)} < +\infty. \end{aligned} \tag{3}$$

It is not difficult to see that the function b is bounded because φ is. Then (3) means

$$\sum_{n \geqslant 0} \left\| \frac{\lambda_n}{b^{\frac{3}{2}}} v_n \right\|^2_{L^2(0,T)} + \sum_{n \geqslant 0} \left\| b^{\frac{1}{2}} v_n' \right\|^2_{L^2(0,T)} < +\infty. \tag{4}$$

In addition, v_n defined by (1) is a solution of $b^2 v_n' + \lambda_n v_n = h_n$. This shows that the condition $\sum_{n \geqslant 0} \left\| b^{\frac{1}{2}} v_n' \right\|^2_{L^2(0,T)} < +\infty$ appearing in (4) follows from (1) and the condition

$$\sum_{n \geqslant 0} \left\| \frac{\lambda_n}{b^{\frac{3}{2}}} v_n \right\|^2_{L^2(0,T)} < +\infty. \tag{5}$$

To complete the proof of Proposition 1, it suffices to prove that (2) leads to (5). To this end, denote by $K(t, s, \lambda_n)$ the following kernel

$$K(t, s, \lambda_n) = \begin{cases} 0 : s \geq t \\ b^{-\frac{1}{2}}(s).e^{-\lambda_n \int_s^t \frac{dr}{b^2(r)}} : s < t. \end{cases}$$

Then, relationship (1) can be written as $v_n(t) = \int_0^T f_n(s) K(t, s, \lambda_n) ds$.

We need the following classical result, the so-called Schur's Lemma.

LEMMA 3. If there exists a constant C such that
a) $\left| \int_0^T \lambda_n b^{-\frac{3}{2}} K(t, s, \lambda_n) ds \right| \leq C$ for almost every $t \in]0, T[$,
b) $\left| \int_0^T \lambda_n b^{-\frac{3}{2}} K(t, s, \lambda_n) dt \right| \leq C$ for almost every $s \in]0, T[$,
then

$$\left\| b^{-\frac{3}{2}} \lambda_n v_n \right\|_{L^2(0,T)} \leq C \|f_n\|.$$

Now, we have to check that the conditions a) and b) are satisfied.

- **Condition a)**

Let ψ be an antiderivative of $\frac{1}{b^2}$. Notice that ψ is then an increasing function. Setting $\sigma = \psi(s)$ and $\eta(\sigma) = b^{\frac{3}{2}}(s)$ we obtain

$$\begin{aligned}
0 &\leq \int_0^t b^{-\frac{1}{2}}(s) e^{\lambda_n \psi(s)} ds = \int_{\psi(0)}^{\psi(t)} e^{\lambda_n \sigma} \eta(\sigma) d\sigma \\
&\leq \frac{e^{\lambda_n \psi(t)} b^{\frac{3}{2}}(t)}{\lambda_n} - \frac{1}{\lambda_n} \int_{\psi(0)}^{\psi(t)} e^{\lambda_n \sigma} \eta'(\sigma) d\sigma \\
&\leq \frac{e^{\lambda_n \psi(t)} b^{\frac{3}{2}}(t)}{\lambda_n} + \frac{3L}{2\lambda_n} \int_{\psi(0)}^{\psi(t)} e^{\lambda_n \sigma} \eta(\sigma) d\sigma
\end{aligned}$$

because $\eta'(\sigma) = \frac{3}{2} b'(s) b(s).\eta(\sigma)$ and $|\eta'(\sigma)| \leq \frac{3L}{2} \eta(\sigma)$. Hence

$$\int_0^t b^{-\frac{1}{2}}(s) e^{\lambda_n \psi(s)} ds \leq \frac{2}{2\lambda_n - 3L} e^{\lambda_n \psi(t)} b^{\frac{3}{2}}(t). \tag{6}$$

Since the condition $\pi^2 > \frac{3L}{2}$ leads to $\lambda_n > \lambda_1 = \pi^2 > \frac{3L}{2}$, there exists a constant $C > 0$ such that

$$\frac{2\lambda_n}{2\lambda_n - 3L} \leq \frac{2\lambda_1}{2\lambda_1 - 3L} = C.$$

So, relationship (6) leads us to

$$\begin{aligned}
\left| \int_0^T \lambda_n b^{-\frac{3}{2}} K(t, s, \lambda_n) ds \right| &= \lambda_n b^{-\frac{3}{2}}(t) e^{-\lambda_n \psi(t)} \int_0^t b^{-\frac{1}{2}}(s) e^{\lambda_n \psi(s)} ds \\
&\leq \frac{2\lambda_n}{2\lambda_n - 3L} \\
&\leq C.
\end{aligned}$$

This shows that Condition a) of Lemma 1 holds true.

- **Condition b**)

Setting $\sigma = \psi(t)$ and $\xi(\sigma) = b^{\frac{1}{2}}(t)$, we obtain

$$\xi'(\sigma) = \frac{b(t) b^{'}(t)}{2} \xi(\sigma) \leq \frac{L}{2} \xi(\sigma).$$

Consequently

$$\begin{aligned}
\left| \int_0^T \lambda_n b^{-\frac{3}{2}}(t) K(t, s, \lambda_n) dt \right| &= \lambda_n b^{-\frac{1}{2}}(s) e^{\lambda_n \psi(s)} \int_s^T b^{-\frac{3}{2}}(t) e^{-\lambda_n \psi(t)} dt \\
&= \lambda_n b^{-\frac{1}{2}}(s) e^{\lambda_n \psi(s)} \int_{\psi(s)}^{\psi(T)} e^{-\lambda_n \sigma} \xi(\sigma) d\sigma \\
&\leq 1 + \frac{L}{2} b^{-\frac{1}{2}}(s) . e^{\lambda_n \psi(s)} \int_{\psi(s)}^{\psi(T)} e^{-\lambda_n \sigma} \xi(\sigma) d\sigma.
\end{aligned}$$

It is easy to see that Condition $b)$ is valid thanks to the inequality $\xi'(\sigma) \leq \frac{L}{2}\xi(\sigma)$. Then, Schur's Lemma is proved, that is

$$\left\| b^{-\frac{3}{2}}\lambda_n v_n \right\|_{L^2(0,T)} \leq C \left\| f_n \right\|.$$

This estimate shows that relationship (5) follows from (2). This ends the proof of Proposition 1.

PROPOSITION 4. Assume that there exists $\varepsilon > 0$ such that the functions $\left(\varphi^{1-\epsilon}\varphi_i'\right)_{i=1,2}$ are bounded. Then the operator

$$b^2(t)a(t,y)\partial_y : H_{1,2}(R) \to H(R)$$

is compact.

PROOF. Observe that

$$b^2(t)a(t,y) = -\varphi\frac{y\varphi' + \varphi_1'}{c^2} = -\varphi^\epsilon\left(\frac{y\varphi^{1-\epsilon}\varphi' + \varphi^{1-\epsilon}\varphi_1'}{c^2}\right).$$

So, the hypothesis shows that the expression $\frac{y\varphi^{1-\epsilon}\varphi' + \varphi^{1-\epsilon}\varphi_1'}{c^2}$ is bounded for c lying between two positive constants. Consequently, it is enough to prove that the operator

$$\varphi^\epsilon.\partial_y : H_{1,2}(R) \to H(R)$$

is compact. To this end, consider the following spaces, equipped with the natural norms

$$\begin{aligned} \mathcal{M} &= \left\{w \in H^{1,2}(R) : \varphi^{-2}w,\ \varphi^{-2}\partial_y w,\ \varphi^{-2}\partial_y^2 w \in L^2(R)\right\}, \\ \mathcal{N} &= \left\{u \in H^{\frac{1}{2},1}(R) : \varphi^{-2}u,\ \varphi^{-2}\partial_y u \in L^2(R)\right\}, \end{aligned}$$

where $H^{\frac{1}{2},1}(R)$ is the Sobolev space defined, for instance, in [5]. It is important to know that if $w \in H^{1,2}(R)$ then $\partial_y w \in H^{\frac{1}{2},1}(R)$. Let us consider the mapping

$$\begin{array}{rcl} H_{1,2}(R) & \to & \mathcal{M} \to \mathcal{N} \to L^2(R), \\ v & \hookrightarrow & \varphi^{\frac{1}{2}}v \hookrightarrow \varphi^{\frac{1}{2}}\partial_y v \hookrightarrow \varphi^{\frac{1}{2}}\partial_y v. \end{array}$$

If a sequence $(v_n)_n$ is weakly convergent to 0 in $H_{1,2}(R)$ then, thanks to the continuity of the mapping $v \hookrightarrow \varphi^{\frac{1}{2}}v$ from $H_{1,2}(R)$ into $\mathcal{M}$, the sequence $(\varphi^{\frac{1}{2}}v_n)_n \in \mathcal{M}$ is also weakly convergent to 0 in $\mathcal{M}$. In addition, the properties of the anisotropic Sobolev spaces show that the sequence $(\varphi^{\frac{1}{2}}\partial_y v_n)_n \in \mathcal{N}$ and converges weakly to 0 (in fact the application $\varphi^{\frac{1}{2}}v \hookrightarrow \varphi^{\frac{1}{2}}\partial_y v$ from $\mathcal{M}$ into $\mathcal{N}$ is continuous).

On the other hand, we know that the canonical injection $H^{\frac{1}{2},1}(R) \hookrightarrow L^2(R)$ is compact (recall that the domain R satisfies the continuation property of Besov). Then, the same holds for the canonical injection $\mathcal{N} \hookrightarrow L^2(R)$. This leads to the strong convergence of the sequence $(\varphi^{\frac{1}{2}}\partial_y v_n)_n$ in $L^2(R)$.

Consequently

$$\lim_n \left\| \varphi^{\frac{1}{2}} \partial_y v_n \right\|_{L^2(R)} = 0. \tag{7}$$

By the weak convergence of $(\varphi^{\frac{1}{2}} \partial_y v_n)_n$ in $\mathcal{N}$, there exists a constant $C > 0$ such that

$$\left\| \frac{1}{\varphi^{\frac{3}{2}}} \partial_y v_n \right\|_{L^2(R)} \leq C. \tag{8}$$

Hence, Cauchy-Schwarz inequality proves that the relationships (7) and (8) give the strong convergence of the sequence $(\varphi^{-\frac{1}{2}} \partial_y v_n)_n$ to 0 in $L^2(R)$. Using again (8), we deduce the convergence of the sequence $(\varphi^{-1} \partial_y v_n)_n$ to 0 in $L^2(R)$. By iteration, we obtain the strong convergence of $(\varphi^{-\frac{3}{2}+\epsilon} \partial_y v_n)_n$ to 0 in $L^2(R)$ for all $\epsilon > 0$. Therefore, the sequence $(\varphi^{\epsilon} \partial_y v_n)_n$ is strongly convergent to 0 in $H(R)$ for all $\epsilon > 0$. The proof of Proposition 2 is complete.

THEOREM 5. Assume that

1) $\varphi^2 c'$ bounded,

2) There exists $\epsilon > 0$ such that $\varphi^{1-\epsilon} \varphi_i'$ is bounded for $i = 1, 2$,

3) $\pi^2 > \frac{3L}{2}$ (where $L = \sup |bb'|$).

Then Problem (P_2) admits a (unique) solution $v \in H_{1,2}(R)$.

PROOF. Let $\mathcal{H}_{1,2}(R)$ be the space defined by

$$\mathcal{H}_{1,2}(R) = \left\{ v \in H_{1,2}(R) : v_{|\partial R \setminus \Gamma_T} = 0 \right\},$$

and observe that the hypotheses of Theorem 2 lead to those of Proposition 1 and 2. Then the operator

$$b^2(t).\partial_t - \frac{\varphi(t)\left(y\varphi'(t) + \varphi_1'(t)\right)}{c^2(t)} \partial_y - \partial_y^2 : \mathcal{H}_{1,2}(R) \to H(R)$$

is an isomorphism because it is the sum of an isomorphism and a compact perturbation (see, for example, [1]).

4 The Initial Problem

We now return to our original problem.

PROPOSITION 6. Under the hypotheses of Theorem. 2, for all $f \in L^2(\Omega)$, Problem (P) admits a (unique) solution in $H^{1,2}(\Omega)$.

PROOF. The uniqueness of the solution is easy to check (see [8]). The existence of the solution follows from Theorem 2 thanks to the relationship between Problems (P) and (P_1). Recall that $y = \frac{x - \varphi_1(t)}{\varphi}$ and $f \in L^2(\Omega)$; then the function h defined on R by $h(t, y) = b^2(t) \widetilde{f}(t, y)$ is an element of $H(R)$ (the converse also holds true). So, by Theorem 2, there exists a solution $v \in H_{1,2}(R)$ to Problem (P_1) when the right-hand side of the equation in (P_1) is equal to h. Let $u(t, x) = v(t, y)$. Then it is easy to check :

$$v \in H(R) \Rightarrow \varphi^2 v \in H(R) \Longleftrightarrow u \in L^2(\Omega),$$

$$\partial_y v \in H(R) \Rightarrow \varphi \partial_y v \in H(R) \Longleftrightarrow \partial_x u \in L^2(\Omega),$$
$$\partial_y^2 v \in H(R) \Longleftrightarrow \partial_x^2 u \in L^2(\Omega),$$
$$\varphi^2 \partial_t v \in H(R) \Longleftrightarrow \varphi^{\frac{1}{2}} \partial_t u \in L^2(\Omega),$$
$$\partial_t u = \partial_t v - \frac{y\varphi' + \varphi_1'}{\varphi} \partial_y v =: w(t, y),$$
$$\partial_t u \in L^2(\Omega) \Leftrightarrow \varphi^{\frac{1}{2}} w \in L^2(R),$$

and

$$\varphi^{\frac{1}{2}} w = \partial_t v - \frac{y\varphi' + \varphi_1'}{\varphi^{\frac{1}{2}}} \partial_y v.$$

Regarding the last equality, observe that $\partial_t v$ and $\frac{y\varphi'+\varphi_1'}{\varphi^{\frac{1}{2}}} \partial_y v$ belong to $L^2(R)$ when $v \in H_{1,2}(R)$. Hence $u \in H^{1,2}(\Omega)$.

We remark that Schur's Lemma allows us to treat the same problem in Sobolev spaces built on general Lebesgue spaces L^p because it does not use the inner product of the Hilbert-Lebesgue space L^2 by contrast to the method used in [8]. This question will be developed in a forthcoming work.

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