An Entire Function Sharing One Small Entire Function With Its Derivative*

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Abstract

In this paper, we study the problem of uniqueness of an entire function sharing a small entire function with its derivative. The results in this paper improve a result given by Brück in 1996, and improve a result given by L. Z. Yang in 1999.

1 Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane, We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [3]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r, h)$ any quantity satisfying $S(r, f) = o(T(r, f)) \, (r \to \infty, r \notin E)$. Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM. Let $a(z)$ be a meromorphic function in the complex plane, if $T(r, a) = S(r, f)$, then $a(z)$ is called a small function of $f(z)$. In this paper, we also need the following three definitions.

DEFINITION 1.1. Let f be a nonconstant entire function, the order of f is defined by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r},$$

where, and in the sequel, $M(r, f) = \max_{|z|=r} |f(z)|$.

DEFINITION 1.2. Let f be a nonconstant entire function, the lower order of f is defined by

$$\mu(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} = \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

DEFINITION 1.3. Let f be a nonconstant entire function, the hyper order of f is defined by

$$\nu(f) = \limsup_{r \to \infty} \frac{\log\log T(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log\log\log M(r, f)}{\log r}.$$

In 1977, Rubel and Yang proved the following result.

THEOREM A (see [8]). Let f be a nonconstant entire function. If f and f' share two finite distinct values CM, then $f \equiv f'$.

In 1996, R. Brück proved the following theorem.

THEOREM B (see [1]). Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer. If f and f' share 0 CM, then $f \equiv cf'$ for some finite complex number $c \neq 0$.

First, consider the differential equation

$$f^{(k)} - e^{\alpha} f = 0, \tag{1}$$

where k is a positive integer and α is an entire function.

In this paper, we will prove the following theorem, which improves Theorem B.

THEOREM 1.1 Let f be a nonconstant solution of (1) such that $\nu(f) < \infty$, and let k be a positive integer, then α is a polynomial, and $\nu(f) = \gamma_{\alpha}$, where γ_{α} is the degree of α.

From Theorem 1.1 we get the following two corollaries.

COROLLARY 1.1. Let f be a nonconstant solution of (1) such that $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer, then $f \equiv cf^{(k)}$ for some finite complex number $c \neq 0$.

COROLLARY 1.2. Let f be a nonconstant solution of (1) such that $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer, and let b be a finite nonzero complex number. If f and $f^{(k)}$ share b IM, then $f \equiv f^{(k)}$.

PROOF. Since f and $f^{(k)}$ share the value b IM, by Hayman's inequality (see [3, Theorem 3.5]) we see that there exists a point z_0 such that $f(z_0) = f^{(k)}(z_0) \neq 0$, which and Corollary 1.1 reveal the conclusion of Corollary 1.2.

In 1996, Brück made the following conjecture.

CONJECTURE 1.1 (see [1]). Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer. If f and f' share one finite value a CM, then $f - a \equiv c(f' - a)$ for some finite complex number $c \neq 0$.

Second, consider the differential equation

$$F^{(k)} - e^{Q(z)} F = 1, \tag{2}$$

where k is a positive integer, and $Q(z)$ is an entire function.

In 1999, L. Z. Yang proved the following result.

THEOREM C (see [10, Theorem 1]). Let $Q(z)$ be a nonconstant polynomial and k be a positive integer. Then every solution F of the differential equation (2) is an entire function of infinite order.

In this paper, we will prove the following result, which improves Theorem C.

THEOREM 1.2. Let $Q(z)$ be a nonconstant polynomial and k be a positive integer. If f is a nonconstant solution of the differential equation

$$f^{(k)} - a = (f - a) \cdot e^{Q(z)}, \tag{3}$$

where $a(z)$ is a small entire function of f such that $\sigma(a) < \gamma_Q$, where γ_Q is the degree of $Q(z)$, then $\nu(f) = \gamma_Q$, and f is an entire function of infinite order.

In the same paper, L. Z. Yang proved the following theorem.

THEOREM D (see [10, Theorem 2]). Let f be a nonconstant entire function of finite order, and let $a(\neq 0)$ be a finite complex number. If f and $f^{(k)}$ share a CM, where k is a positive integer, then $f - a \equiv c(f^{(k)} - a)$ for some finite complex number $c \neq 0$.

From Theorem 1.2 we get the following result, which improves Theorem D.

THEOREM 1.3. Let f be a nonconstant entire function of finite order, and let $a(z)$ be a small entire function of f, such that $\sigma(a) < 1$. If $f - a$ and $f^{(k)} - a$ share 0 CM, where k is a positive integer, then $f - a \equiv c(f^{(k)} - a)$ for some finite complex number $c \neq 0$.

From Theorem 1.3 we get the following two corollaries.

COROLLARY 1.3. Let f be a nonconstant entire function of finite order. If f and $f^{(k)}$ have the same fixed points with the same multiplicities, where k is a positive integer, then $f - z \equiv c(f^{(k)} - z)$ for some finite complex number $c \neq 0$.

COROLLARY 1.4. Let f be a nonconstant entire function of finite order, and let $a(z)$ be a small entire function of f, such that $\sigma(a) < 1$, and let b be a finite complex number such that $a \not\equiv b$. If $f - a$ and $f^{(k)} - a$ share 0 CM, and if $f - b$ and $f^{(k)} - b$ share 0 IM, where k is a positive integer, then $f \equiv f^{(k)}$.

PROOF. First, from Theorem 1.3 we see that there exists some finite nonzero complex number c such that $f - a \equiv c(f^{(k)} - a)$, which implies that $T(r, f) = T(r, f^{(k)}) + S(r, f)$. From this and Lemma 2.6 in Section 2 of this paper we get the conclusion of Corollary 1.4.

2 Some Lemmas

We state several preparatory Lemmas.

LEMMA 2.1 (see [5, Corollary 2.3.4] or [9, Lemma 1.4]). Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then $m(r, f^{(k)}/f) = O(\log(rT(r, f)))$, outside of a possible exceptional set E of finite linear measure, and if f is of finite order of growth, then $m(r, f^{(k)}/f) = O(\log r)$.

LEMMA 2.2 (see [5, Lemma 1.1.1]). Let $g : (0, +\infty) \longrightarrow R$, $h : (0, +\infty) \longrightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

LEMMA 2.3 (see [2, Lemma 2]). If f is a transcendental entire function of hyper order $\nu(f)$, then

$$\nu(f) = \limsup_{r \to \infty} \frac{\log\log\nu(r, f)}{\log r},$$

where, and in the sequel, $\nu(r, f)$ denotes the central-index of $f(z)$.

LEMMA 2.4 (see [5, Proposition 8.1]). Let

$$w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_0(z)w = F(z) \tag{4}$$

be a non-homogeneous linear differential equation with entire coefficients $a_0(z)(\not\equiv 0)$, $a_1(z), \cdots a_{n-1}(z)$ and $F(z)(\not\equiv 0)$. Then all solutions of (4) are entire functions.

LEMMA 2.5 (see [4, pp. 36–37] or [5, Theorem 3.1]). If f is an entire function of order $\sigma(f)$, then

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log\nu(r, f)}{\log r}.$$

LEMMA 2.6 (see [7, Proof of Lemma 3]). Let f be a nonconstant entire function, and let $a\,(\not\equiv \infty)$ and $b\,(\not\equiv \infty)$ be two distinct small functions of f. If f and $f^{(k)}$ share a and b IM, where k is a positive integer, and if $T(r, f) = T(r, f^{(k)}) + S(r, f)$, then $f \equiv f^{(k)}$.

3   Proof of Theorems

We now prove our main results.

PROOF OF THEOREM 1.1. Since f is a nonconstant entire function, from (1) we deduce that f is a transcendental entire function. Again from (1) and Lemma 2.1 we deduce

$$T(r, e^\alpha) = O(\log r T(r, f))\,(r \notin E). \tag{5}$$

From (5) and Lemma 2.2 we see that there exists a sufficiently large positive number r_0 such that

$$T(r, e^\alpha) = O(\log 2r + \log T(2r, f))\,(r \geq r_0). \tag{6}$$

Since $\nu(f) < \infty$, from (6) we deduce

$$\sigma(e^\alpha) \leq \nu(f) < \infty, \tag{7}$$

which implies that α is a polynomial and that

$$\sigma(e^\alpha) = \gamma_\alpha, \tag{8}$$

where γ_α is the degree of α. On the other hand, from the condition that f is a nonconstant entire function, we have

$$M(r, f) \to \infty, \tag{9}$$

as $r \to \infty$. Again let

$$M(r, f) = |f(z_r)|, \tag{10}$$

where $z_r = re^{i\theta(r)}$, and $\theta(r) \in [0, 2\pi)$ is some nonnegative real number. From (10) and the Wiman-Valiron theory (see [5, Theorem 3.2]), we see that there exists a subset $E_1 \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_1} \frac{dt}{t} < \infty$, such that for some point $z_r = re^{i\theta(r)}$ $(\theta(r) \in [0, 2\pi))$ satisfying $|z_r| = r \notin E_1$ and $M(r, f) = |f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)), \tag{11}$$

as $r \to \infty$. Substituting (9)-(11) into (1) we get

$$\left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)) = e^{\alpha(z_r)}, \tag{12}$$

as $r \to \infty$. From (12) we get

$$\limsup_{r \to \infty} \frac{\log \log \nu(r, f)}{\log r} \leq \limsup_{r \to \infty} \frac{\log \log(\frac{(\nu(r,f))^k}{|z_r|^k} \cdot |1 + o(1)|)}{\log r} \leq \limsup_{r \to \infty} \frac{\log \log M(r, e^\alpha)}{\log r},$$

and so it follows from Lemma 2.3 that $\nu(f) \leq \sigma(e^\alpha)$. From this and (7)-(8) we get the conclusion of Theorem 1.1.

PROOF OF THEOREM 1.2. From (3) and Lemma 2.4 we see that every solution of the equation (3) is a transcendental entire function. Noting that a is a small entire function of f, from (3) and Lemma 2.1 we can deduce

$$T(r, e^Q) \leq (k + 3)T(r, f) + O(\log r T(r, f)) \, (r \notin E). \tag{13}$$

From (13) and Lemma 2.2 we see that there exists a sufficiently large positive number r_0, such that

$$T(r, e^Q) \leq (k + 3)T(2r, f) + O(\log 2r + \log T(2r, f)) \, (r \geq r_0), \tag{14}$$

From (14) we deduce $\mu(e^Q) \leq \mu(f)$. Combining $\mu(e^Q) = \sigma(e^Q) = \gamma_Q \geq 1$ and $\sigma(a) < 1$, we get

$$\mu(f) > \sigma(a). \tag{15}$$

Let

$$Q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1 z + q_0, \tag{16}$$

where $q_n (\neq 0), q_{n-1}, \cdots, q_1$ and q_0 are finite complex numbers. Then from (16) we get

$$\lim_{|z| \to \infty} \frac{Q(z)}{q_n z^n} = 1. \tag{17}$$

From (17) we see that there exists some sufficiently large positive number r_0, such that

$$\frac{|Q(z)|}{|q_n z^n|} > \frac{1}{e} \quad (|z| > r_0). \tag{18}$$

From (3) and (18) we deduce

$$
\begin{aligned}
n \log r + \log |q_n| - 1 &= \log \frac{|q_n z^n|}{e} \le \log |Q| = \log |\log e^Q| \le |\log \log e^Q| \\
&= |\log \log \frac{f^{(k)} - a}{f - a}| \quad (|z| > r_0),
\end{aligned}
$$

namely

$$
n \log r + \log |q_n| - 1 \le |\log \log \frac{f^{(k)} - a}{f - a}| \quad (|z| > r_0). \tag{19}
$$

From the condition that f is a nonconstant entire function we have (9)-(11). Since

$$
\frac{f^{(k)} - a}{f - a} = \frac{\frac{f^{(k)}}{f} - \frac{a}{f}}{1 - \frac{a}{f}}. \tag{20}
$$

From (9)-(11), (15) and the Definitions 1.1 and 1.2, we get

$$
\frac{a(z_r)}{f(z_r)} \longrightarrow 0, \tag{21}
$$

as $|z_r| \to \infty$. Thus from (11), (19)-(21) we deduce

$$
n \log |z_r| + \log |q_n| - 1 \le |\log \log ((\frac{\nu(r, f)}{z_r})^k (1 + o(1)))| \tag{22}
$$

and

$$
\log ((\frac{\nu(r, f)}{z_r})^k (1 + o(1))) = k(\log \nu(r, f) - \log r - i\theta(r)) + o(1), \tag{23}
$$

as $r \to \infty$. We discuss the following two cases.

CASE 1. Suppose that

$$
\limsup_{r \to \infty} \frac{\log \nu(r, f)}{\log r} = \infty. \tag{24}
$$

Noting that $\theta(r) \in [0, 2\pi)$, from (23), (24) and Lemma 2.3 we have

$$
\begin{aligned}
&\limsup_{r \to \infty} \frac{\left| \log \log \left(\left(\frac{\nu(r,f)}{z_r} \right)^k (1 + o(1)) \right) \right|}{\log r} \\
&\le \limsup_{r \to \infty} \frac{\log \log \nu(r, f)}{\log r} + \limsup_{r \to \infty} \frac{\left| \log \left(1 - \frac{\log r}{\log \nu(r,f)} - \frac{i\theta(r)}{\log \nu(r,f)} \right) \right|}{\log r} \\
&+ \lim_{r \to \infty} \frac{\log 2}{\log r} + \lim_{r \to \infty} \frac{2k_1 \pi}{\log r} + \lim_{r \to \infty} \frac{k}{\log r} \\
&= \limsup_{r \to \infty} \frac{\log \log \nu(r, f)}{\log r} = \nu(f),
\end{aligned}
$$

where k_1 is some nonnegative integer. Combining (22) and the condition $|z_r| = r$ we deduce

$$n \leq \limsup_{r \to \infty} \frac{\log \log \nu(r, f)}{\log r} = \nu(f). \tag{25}$$

From (16) we have

$$\sigma(e^Q) = \gamma_{Q(z)} = n. \tag{26}$$

From (25) and (26) we get

$$\sigma(e^Q) \leq \nu(f). \tag{27}$$

On the other hand, from (3), (9)-(11), (20) and (21) we get

$$\left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)) = e^{Q(z_r)}, \tag{28}$$

as $r \longrightarrow \infty$. From (24) and (28) we deduce

$$
\begin{aligned}
\limsup_{r \to \infty} \frac{\log \log \nu(r, f)}{\log r} &= \limsup_{r \to \infty} \frac{\log \log \left(\frac{\nu(r,f)}{2r}\right)^k}{\log r} \\
&\leq \limsup_{r \to \infty} \frac{\log \log \left(\left(\frac{\nu(r,f)}{|z_r|}\right)^k \cdot |1 + o(1)|\right)}{\log r} \\
&\leq \limsup_{r \to \infty} \frac{\log \log M(r, e^{Q(z)})}{\log r}.
\end{aligned}
$$

From this, Lemma 2.3 and the definition of the order of an entire function we get

$$\nu(f) \leq \sigma(e^Q). \tag{29}$$

From (26), (27) and (29) we can get the conclusion of Theorem 1.1.

CASE 2. Suppose that

$$\limsup_{r \to \infty} \frac{\log \nu(r, f)}{\log r} < \infty. \tag{30}$$

First, from (30) we deduce

$$\nu(f) = 0. \tag{31}$$

On the other hand, from (3), (9)-(11), (20), (21), (23) and Lemma 2.5 we deduce $|Q(z_r)| = |\log e^{Q(z_r)}| = |k(\log \nu(r, f) - \log r - i\theta(r)) + o(1)| \leq O(\log r)$, as $r \to \infty$. Combining the condition that $Q(z)$ is a polynomial, we can deduce $Q(z)$ is a constant, and so $\gamma_Q = 0$. From this and (31) we get the conclusion of Theorem 1.2.

Theorem 1.2 is thus completely proved.

PROOF OF THEOREM 1.3.   From the assumptions of Theorem 1.3 we have

$$f^{(k)} - a = (f - a) \cdot e^\beta, \tag{32}$$

where β is a polynomial. If β is a constant, then the conclusion of Theorem is obvious. Next we suppose that β is not a constant, and so $\gamma_\beta \geq 1$, where γ_β is the degree of β.

Since $\sigma(a) < 1$, thus $\sigma(a) < \gamma_\beta$. Combining (32) and Theorem 1.2 we deduce that f is an entire function of infinite order, which contradicts the assumption of Theorem 1.3. Theorem 1.3 is thus completely proved.

References

[1] R. Brück, On entire functions which share one value CM with their first derivative, Results in Math., 30(1996), 21–24.

[2] Z. X. Chen and C. C. Yang, Some further results on the zeros and growths of entire solutions of second order linear differential equation, Kodai Math J., 22(1999), 273–285.

[3] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[4] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel-Boston, 1985.

[5] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[6] A. I. Markushshevich, Theory of Functions of a Complex Variable, Volume 2, translated by R. Silverman, Prenctice-Hall, Englewood Cliffs, 1965.

[7] G. D. Qiu, Uniqueness of entire functions that share some small functions, Kodai Math. J., 23(2000), 1-11.

[8] L. Rubel and C. C. Yang, Values shared by an entire function and its derivative, in "Complex Analysis, Kentucky 1976", Lecture Notes in Mathematics, Vol 599, pp. 101–103, Springer-Verlag, Berlin, 1977.

[9] C. C. Yang and H. X. Yi,, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Press, Dordrecht/Boston/London, 2003.

[10] L. Z. Yang, Solution of a differential equation and its applications, Kodai Math J., 22(1999), 458–464.