Applied Mathematics E-Notes, 8(2008), 231-237 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/~amen/

On Nonunique Fixed Point Theorems*

Zeqing Liu†, Li Wang‡, Shin Min Kang§, Yong Soo Kim§

Received 9 April 2007

Abstract

Several fixed point theorems for four classes of mappings in complete metric spaces are given. The results presented in this paper extend properly the Banach contraction principle.

1 Introduction and Preliminaries

Let (X, d) be a metric space, $T : X \to X$ be a mapping, and $r \in [0, 1)$ be a constant. Let $\mathbb{N}$ denote the set of all positive integers. A point $x_0 \in X$ is called an *n-periodic point* of T, if there exists $n \in \mathbb{N}$ such that $x_0 = T^n x_0$ but $x_0 \neq T^k x_0$ for $k = 1, 2, 3, ..., n-1$. For $x \in X$, the set $O_T(x) = \{T^n x : n \geq 0\}$ is said to be the *orbit* of T at x. Let Φ be the set of $\phi : [0, \infty) \to [0, \infty)$ which is nondecreasing and $\phi(t) < t$ and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

It is easy to see that $\phi(t) = rt \in \Phi$ for any constant $r \in (0, 1)$. Rhoades [1] provided some fixed point theorems for various contractive mappings.

In this paper, we will discuss the existence of fixed points for mappings T that satisfy

$$d(Tx, Ty) + d(Ty, Tz) \leq \phi(d(x, y) + d(y, z)) \tag{1}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$, where $\phi \in \Phi$; or

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \phi(d(x, y) + d(y, z) + d(z, x)) \tag{2}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$, where $\phi \in \Phi$; or

$$\max\{d(Tx, Ty), d(Ty, Tz)\} \leq \phi(\max\{d(x, y), d(y, z)\}) \tag{3}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$, where $\phi \in \Phi$; or

$$\max\{d(Tx, Ty), d(Ty, Tz), d(Tz, Tx)\} \leq \phi(\max\{d(x, y), d(y, z), d(z, x)\}) \tag{4}$$

*Mathematics Subject Classifications: 54H25

†Department of Mathematics, Liaoning Normal University, P. O. Box 200, Dalian, Liaoning 116029, P. R. China

‡Department of Science, Shenyang Institute of Aeronautical Engineering, Shenyang, Liaoning 110034, P. R. China

§Department of Mathematics and the Research Institute of Natural Science, Gyeongsang National University, Jinju 660-701, Korea

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$, where $\phi \in \Phi$.

It follows from the definition of n-periodic point that

LEMMA 1. Let T be a mapping from a metric space (X, d) into itself. If $x_0 \in X$ is an n-periodic point of T, then $T^i x_0 \neq T^j x_0$ for all $0 \leq i < j \leq n-1$.

2 Main Results

Our first main result is the following.

THEOREM 1. Let (X, d) be a complete metric space and $T : X \to X$ satisfy (1). Then

(a) T has at most two distinct fixed points in X;
(b) if T has 2-periodic points in X, then they are exactly two;
(c) T has no any n-periodic points in X for $n \geq 3$;
(d) T has a fixed point in X provided that T has an orbit without 2-periodic points.

PROOF. First, we assert that T has at most two distinct fixed points in X. Otherwise T has (at least) three different fixed points a, b, c in X. In the light of (1), we infer that

$$\begin{aligned} d(a,c) + d(c,b) &= d(Ta, Tc) + d(Tc, Tb) \\ &\leq \phi(d(a,c) + d(c,b)) \\ &< d(a,c) + d(c,b), \end{aligned}$$

which is a contradiction.

Suppose that there exists a point $b \in X$ which is a 2-periodic point of T. Then Tb is also a 2-periodic point of T different from b. Now we claim that T has the only two 2-periodic points b and Tb. Otherwise there is a point $c \in X$ which is also a 2-periodic point of T with $b \neq c \neq Tb$. It is easy to show that $Tb \neq Tc \neq T^2 b \neq Tb$. By (3) we have

$$\begin{aligned} d(b,c) + d(c,Tb) &= d(T^2 b, T^2 c) + d(T^2 c, T^3 b) \\ &\leq \phi(d(Tb, Tc) + d(Tc, T^2 b)) \\ &= \phi(d(T^3 b, T^3 c) + d(T^3 c, T^2 b)) \\ &\leq \phi^2(d(T^2 b, T^2 c) + d(T^2 c, Tb)) \\ &< d(b,c) + d(c, Tb), \end{aligned}$$

which is a contradiction. Thus T has only 2-periodic points b and Tb.

Now we exclude the presence of n-periodic point for $n \geq 3$. Suppose that $a_0 \in X$ is an n-periodic point of T for $n \geq 3$. Let $a_k = T^k a_0$, $d_k = d(a_k, a_{k+1}) + d(a_{k+1}, a_{k+2})$ for all $0 \leq k \leq n$. From Lemma 1 and (1), we have

$$\begin{aligned} d_k &= d(Ta_{k-1}, Ta_k) + d(Ta_k, Ta_{k+1}) \\ &\leq \phi(d(a_{k-1}, a_k) + d(a_k, a_{k+1})) \\ &= \phi(d_{k-1}) < d_{k-1} \end{aligned} \tag{5}$$

for all $1 \leq k \leq n$. In view of (3) and (5), we get that

$$d_0 = d_n \leq \phi(d_{n-1}) < d_{n-1} \leq \cdots < d_0,$$

which is a contradiction.

Suppose that there exists a point $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Set $x_n = T^n x_0$, $d_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$ for any $n \geq 0$. If there exists some $n \geq 0$ with $x_n = x_{n+1}$, then x_n is a fixed point of T; if $x_n \neq x_{n+1}$ for any $n \geq 0$, from (1) we have

$$d_n \leq \phi(d_{n-1}) \leq \phi^2(d_{n-2}) \leq \cdots \leq \phi^n(d_0). \tag{6}$$

For each $r, s, m \in \mathbb{N}$ with $r > s \geq m$, by the triangular inequality and (6), we get that

$$d(x_r, x_s) \leq \sum_{n=m}^{r-1} d_n \leq \sum_{n=m}^{r-1} \phi^n(d_0). \tag{7}$$

Since $\phi \in \Phi$, (7) ensures that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in X. It follows from completeness of (X, d) that there exists a point $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Obviously, there exists some integer $k \in \mathbb{N}$ with $x_n \neq a$ for all $n \geq k$. From (c) and (5) we obtain that

$$\begin{aligned} d(x_{n+1}, Ta) + d(Ta, Tx_{n+2}) &\leq \phi(d(x_n, a) + d(a, x_{n+2})) \\ &< d(x_n, a) + d(a, x_{n+2}) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, which implies that $\lim_{n\to\infty} x_n = Ta$. Hence $Ta = a$. This completes the proof.

The proof of the next result is similar to that of Theorem 1 and is omitted.

THEOREM 2. Let (X, d) be a complete metric space and $T : X \to X$ satisfy (2). Then the conclusions of Theorem 1 hold.

THEOREM 3. Let (X, d) be a complete metric space and $T : X \to X$ satisfy (3). Then the conclusions of Theorem 1 hold.

PROOF. We first assert that T has at most two distinct fixed points in X. Otherwise T has three different fixed points a, b, c in X. From (3), we obtain that

$$\begin{aligned} \max\{d(a, b), d(b, c)\} &= \max\{d(Ta, Tb), d(Tb, Tc)\} \\ &\leq \phi(\max\{d(a, b), d(b, c)\}) \\ &< \max\{d(a, b), d(b, c)\}, \end{aligned}$$

which is a contradiction.

Suppose that T has a 2-periodic point $b \in X$. Then Tb is also a 2-periodic point of T different from b. We point out that b and Tb are the only two 2-periodic points of T in X. Otherwise there exists c in X which is also a 2-periodic point of T and

$b \neq c \neq Tb$. By (3) we get that

$$\begin{aligned}\max\{d(Tb,Tc),d(Tc,b)\} &= \max\{d(Tb,Tc),d(Tc,T^2b)\}\\ &\leq \phi(\max\{d(b,c),d(c,Tb)\})\\ &= \phi(\max\{d(T^2b,T^2c),d(T^2c,T^3b)\})\\ &\leq \phi^2(\max\{d(Tb,Tc),d(Tc,b)\})\\ &< \max\{d(Tb,Tc),d(Tc,b)\},\end{aligned}$$

which is impossible.

We next conclude that T has no n-periodic point for $n \geq 3$. Suppose that T has an n-periodic point a_0 for $n \geq 3$. Set $a_k = T^k a_0$, $d_k = d(a_k, a_{k+1})$ for all $0 \leq k \leq n$. According to Lemma 1 and (3), we get that

$$\begin{aligned}\max\{d_0,d_1\} &= \max\{d_n,d_{n+1}\}\\ &= \max\{d(Ta_{n-1},Ta_n),d(Ta_n,Ta_{n+1})\}\\ &\leq \phi(\max\{d(a_{n-1},a_n),d(a_n,a_{n+1})\})\\ &= \phi(\max\{d_{n-1},d_n\}) \leq \phi^n(\max\{d_0,d_1\})\\ &< \max\{d_0,d_1\},\end{aligned}$$

which is a contradiction.

Lastly, we prove that T has a fixed point in X provided that T has an orbit without 2-periodic points in X. Assume that there exists a point $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Let $x_n = T^n x_0$, $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$. We consider two cases:

Case 1. There exists some $n \geq 0$ with $x_n = x_{n+1}$. Then x_n is a fixed point of T in X.

Case 2. For all $n \geq 0$, $x_n \neq x_{n+1}$. It follows that $x_n \neq x_m$ for $n > m \geq 0$. In view of (3) we have

$$\begin{aligned}\max\{d_n,d_{n+1}\} &= \max\{d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1})\}\\ &\leq \phi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\\ &= \phi(\max\{d_{n-1},d_n\}) \leq \phi^2(\max\{d_{n-2},d_{n-1}\})\\ &\leq \cdots \leq \phi^n(\max\{d_0,d_1\}).\end{aligned}$$

For each $n \in \mathbb{N}$ and $p \in \mathbb{N}$, using the triangular inequality and (3), we obtain that

$$d(x_n,x_{n+p}) \leq \sum_{i=n}^{n+p-1} d_i \leq \sum_{i=n}^{n+p-1} \phi^i(\max\{d_0,d_1\}),$$

which yields that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. It follows from the completeness of (X,d) that $\lim_{n\to\infty} x_n = a$ for some point $a \in X$. It is easy to check that there exists some integer $k \geq 1$ with $x_n \neq a$ for all $n \geq k$. Using again (3) we have

$$\begin{aligned}\max\{d(x_{n+1},Ta),d(Ta,x_{n+2})\} &\leq \phi(\max\{d(x_n,a),d(a,x_{n+1})\})\\ &< \max\{d(x_n,a),d(a,x_{n+1})\}\\ &\to 0\end{aligned}$$

as $n \to \infty$, that is, $Ta = a$. This completes the proof.

REMARK 1. In Theorem 3, the presence of 2-periodic points excludes the presence of fixed points and vice verse. Otherwise there exist two points $a, b \in X$ such that $a = Ta, b = T^2 b$ with $a \neq Tb \neq b$. In view of (3), we obtain that

$$\begin{aligned}\max\{d(a,b), d(b,Tb)\} &= \max\{d(T^2a, T^2b), d(T^2b, T^3b)\}\\ &\leq \phi(\max\{d(Ta,Tb), d(Tb,T^2b)\})\\ &\leq \phi^2(\max\{d(a,b), d(b,Tb)\})\\ &< \max\{d(a,b), d(b,Tb)\},\end{aligned}$$

which is a contradiction.

REMARK 2. Theorem 3 extends properly the Banach contraction principle.

Now we give the following examples for Remarks 1 and 2.

EXAMPLE 1. Let $X = \{1,2,3,4\}$, $T : X \to X$ be a mapping defined by $T1 = 1$, $T2 = 2$, $T3 = 4$, $T4 = 2$ and $d : X \times X \to [0,\infty)$ be a function defined by $d(1,2) = 1$, $d(2,3) = 3$, $d(1,3) = 4$, $d(2,4) = 2$, $d(1,4) = 2.5$, $d(3,4) = 3.5$, $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x, y \in X$. Take $\phi(t) = \frac{3}{4}t$ for $t \geq 0$. It is easy to check that the conditions of Theorem 3 are satisfied, and T has two fixed points 1 and 2 in X. But the Banach contraction principle is not available and T has no 2-periodic points in X.

EXAMPLE 2. Let $X = \{1,2,3\}$, $T : X \to X$ be a mapping defined by $T1 = 2$, $T2 = 1$, $T3 = 2$ and $d : X \times X \to [0,\infty)$ be a function defined by $d(1,2) = 3$, $d(1,3) = 4$, $d(2,3) = 5$, $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x, y \in X$. Put $\phi(t) = \frac{4}{5}t$ for $t \geq 0$. Clearly, the conditions of Theorem 3 are satisfied and T has two 2-periodic points 1 and 2 in X, but T has no fixed points in X.

THEOREM 4. Let (X,d) be a complete metric space and $T : X \to X$ satisfy (4). Then the conclusions of Theorem 1 hold.

PROOF. First we claim that T has at most two distinct fixed points in X. Otherwise there are three different points a, b, c in X, which are all fixed points of T. By (4) we get that

$$\begin{aligned}\max\{d(a,b), d(b,c), d(c,a)\} &= \max\{d(Ta,Tb), d(Tb,Tc), d(Tc,Ta)\}\\ &\leq \phi(\max\{d(a,b), d(b,c), d(c,a)\})\\ &< \max\{d(a,b), d(b,c), d(c,a)\},\end{aligned}$$

which is a contradiction.

We next assert that if T has 2-periodic point $b \in X$, then b and Tb are all 2-periodic points of T. Otherwise, there exists a point c in X which is a 2-periodic point with $b \neq c \neq Tb$. By (4) we have

$$\begin{aligned}\max\{d(b,Tb), d(Tb,c), d(c,b)\} &= \max\{d(T^2b, T^3b), d(T^3b, T^2c), d(T^2c, T^2b)\}\\ &\leq \phi(\max\{d(Tb, T^2b), d(T^2b, Tc), d(Tc, Tb)\})\\ &\leq \phi^2(\max\{d(b,Tb), d(Tb,c), d(c,b)\}),\\ &< \max\{d(b,Tb), d(Tb,c), d(c,b)\},\end{aligned}$$

which is impossible.

We now exclude the presence of n-periodic point for $n \geq 3$. Suppose that T has an n-periodic point $a_0 \in X$ for $n \geq 3$. Set $a_k = T^k a$ for all $0 \leq k \leq n$. According to (4), we know that

$$\begin{aligned}
&\max\{d(a_0, a_1), d(a_1, a_2), d(a_2, d_0)\} \\
&= \phi(\max\{d(Ta_{n-1}, Ta_n), d(Ta_n, Ta_{n+1}), d(Ta_{n+1}, Ta_{n-1})\}) \\
&\leq \phi(\max\{d(a_{n-1}, a_n), d(a_n, a_{n+1}), d(a_{n+1}, a_{n-1})\}) \\
&\leq \phi^2(\max\{d(a_{n-2}, a_{n-1}), d(a_{n-1}, a_n), d(a_n, a_{n-2})\}) \\
&\leq \cdots \\
&\leq \phi^n(\max\{d(a_0, a_1), d(a_1, a_2), d(a_2, a_0)\}) \\
&< \max\{d(a_0, a_1), d(a_1, a_2), d(a_2, a_0)\},
\end{aligned}$$

which is a contradiction.

Finally we assert that T has a fixed point in X provided that T has an orbit without 2-periodic points. Suppose that there exists a point $x_0 \in X$ and $O_T(x_0)$ is such an orbit that T has no 2-periodic points in it. Let $x_n = T^n x_0$ for all $n \geq 0$. We have to consider the following two cases:

Case 1.There exists some $n \geq 0$ with $x_n = x_{n+1}$. Then x_n is a fixed point of T in X.

Case 2. For all $n \geq 0$, $x_n \neq x_{n+1}$. Then $x_n \neq x_m$ for all $n > m \geq 0$. In view of (4), we have

$$\begin{aligned}
&\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_n)\} \\
&= \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n-1})\} \\
&\leq \phi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_{n-1})\}) \\
&\leq \phi^2(\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n-2})\}) \\
&\leq \cdots \\
&\leq \phi^n(\max\{d(x_0, x_1), d(x_1, x_2), d(x_2, x_0)\}),
\end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq \phi^n(\max\{d(x_0, x_1), d(x_1, x_2), d(x_2, x_0)\}).$$

By the triangular inequality and (4), we obtain that

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}) \\
&\leq \sum_{i=n}^{n+p-1} \phi^i(\max\{d(x_0, x_1), d(x_1, x_2), d(x_2, x_0)\})
\end{aligned}$$

for all $n, p \in \mathbb{N}$. Clearly, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence and hence $\lim_{n\to\infty} x_n = a$ for some $a \in X$ since (X, d) is complete. Obviously, there exists some integer $k \geq 1$ with

$x_n \neq a$ for all $n \geq k$. Hence we have

$$\begin{aligned}
&\max\{d(Ta, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, Ta)\} \\
&\leq \phi(\max\{d(a, x_n), d(x_n, x_{n+1}), d(x_{n+1}, a)\}) \\
&< \max\{d(a, x_n), d(x_n, x_{n+1}), d(x_{n+1}, a)\} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. That is, $\lim_{n\to\infty} x_n = Ta$. Thus $Ta = a$. This completes the proof.

REMARK 3. The following example reveals that Theorem 4 extends indeed the Banach contraction principle.

EXAMPLE 3. Let X, T and d be as in Example 1. Put $\phi(t) = \frac{2}{3}t$. Then it is easy to verify that the conditions of Theorem 4 are fulfilled, and T has two fixed points 1 and 2. But the Banach contraction principle is not applicable.

References

[1] B. E. Rhoades, A comparison of various definitions of contraction mappings, Trans. Amer. Math. Soc., 226(1997), 257–289.