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Rate Of Convergence Of Chlodowsky Operators For Functions With Derivatives Of Bounded Variation*

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Abstract

In the present paper, we estimate the rate of pointwise convergence of the Chlodowsky operators C_n for functions, defined on the interval $[0, b_n]$ with derivatives of bounded variation, where $\lim_{n\to\infty} b_n = \infty$.

1 Introduction

For a function defined on the interval $[0, 1]$, the classical Bernstein operators are

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x), \;\; 0 \le x \le 1, \quad n \ge 1, \tag{1}$$

where $p_{k,n}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k}$ is the Bernstein basis. Bernstein [1] used these operators (1) to give the first constructive proof of the Weierstrass theorem. It is well known that if we make the substitution $x = \frac{x}{b_n}$ and replacing the discrete values $f(\frac{k}{n})$ by $f(\frac{k}{n}b_n)$, in the polynomial of Bernstein $B_n(f;x)$ corresponding to a function f defined on $[0, 1]$, then one can obtain the following polynomials

$$C_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) \left(\begin{array}{c} n \\ k \end{array}\right) \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, 0 \le x \le b_n \tag{2}$$

where (b_n) is a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

These operators are called Bernstein-Chlodowsky operators or Chlodowsky operators. Operators of type (2) were introduced by Chlodowsky [2] and further modified and studied by many authors [3-4]. Since the behaviour of Chlodowsky operators are very

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similar to the Bernstein operators, these operators allow us to investigate approximation properties of functions defined on the infinite interval $0 \leq x < \infty$ by using the similar techniques and methods on clasical and modified Bernstein operators. For example, in [4] Ibikli and Karsli approximated integrable functions on the interval $[0, b_n]$ by what they called "Chlodowsky Type Durrmeyer operators" defined as follows: $D_n : BV[0,\infty) \to \mathcal{P}$,

$$D_n(f;x) = \frac{(n+1)}{b_n} \sum_{k=0}^{n} p_{k,n}\left(\frac{x}{b_n}\right) \int_0^{b_n} f(t) p_{k,n}\left(\frac{t}{b_n}\right) dt, \quad 0 \leq x \leq b_n,$$

where $p_{k,n}(x)$ is the Bernstein basis. They estimated the convergence rate of these D_n operators for functions in $BV[0,\infty)$. After this study Karsli [5] obtained their rate of convergence for functions whose derivatives had bounded variation in $[0,\infty)$. It is useful to mention very recent papers by Karsli and Ibikli [6-7], which deal with the rate of pointwise convergence of the operators (2) and its Bézier variant in the space $BV[0, b_n]$ respectively.

The concern of this paper is to study the rate of convergence of operators C_n to the limit f of functions with derivatives of bounded variation on the interval $[0, b_n]$, $(n \to \infty)$ extending infinity. At the point x, which is a discontinuity of the first kind of the derivative, we shall prove that $C_n(f;x)$ converge to the limit $f(x)$.

Some authors studied some linear positive operators and obtained the rate of convergence for functions in $DBV(I)$. For example, Bojanic and Cheng investigated the rate of convergence of Hermite-Fejer polynomials for functions with derivatives of bounded variation [8] and they also investigated in the paper [9] the asymptotic behavior of Bernstein polynomials for functions in $DBV[0,1]$ of all functions f that can be written as

$$f(x) = f(0) + \int_0^x \Psi(t)dt, \quad x \in [0,1],$$

where $\Psi \in BV[0,1]$. We also mention some recent studies in this area by Gupta et al. [10], in which they estimated the rate of convergence of summation-integral-type operators for functions in γ-weighted space $DBV_\gamma(0,\infty)$, $(\gamma \geq 0)$, and by Gupta et al. [11] and very recent papers by Karsli [12].

Let $DBV(I)$ denote the class of differentiable functions defined on a set $I \subset R$, whose derivatives are bounded variation on I,

$$DBV(I) = \{f : f' \in BV(I)\}.$$

It is clear that the class of functions $DBV(I)$ considered here is much more general than the class of functions with continuous derivative on I.

For the sake of brevity, let the auxiliary function f_x be defined by

$$f_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq b_n \\ 0, & t = x \\ f(t) - f(x-), & 0 \leq t < x \end{cases}.$$

The main theorem of this paper is as follows.

THEOREM. Let f be a function with derivatives of bounded variation on every finite subinterval of $[0,\infty)$ and $\lim_{x\to\infty} f'(x)$ exists. Then for every $x \in (0,\infty)$, we have

$$|C_n(f;x)-f(x)| \leq \left|\frac{f'(x+)-f'(x-)}{2}\right| \sqrt{\frac{x(b_n-x)}{n}} + \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{b_n-x}{k}} (f'_x). \qquad (3)$$

where $\bigvee_a^b (f'_x)$ is the total variation of f'_x on $[a,b]$.

2 Auxiliary Results

In this section we give certain results, which are necessary to prove our main theorems.

As before we let

$$\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) := \int_0^t \frac{\partial}{\partial u} K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right) du$$

where $0 \leq t \leq b_n$, and

$$K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right) = \begin{cases} \sum\limits_{kb_n \leq nu} p_{k,n}(\frac{x}{b_n}), & 0 < u \leq b_n \\ 0, & u = 0 \end{cases}. \qquad (4)$$

Then $\lambda_n(\frac{x}{b_n},\frac{t}{b_n}) \leq 1$. Since the operators (1) and (2) are special cases of Stieltjes integrals, alternatively we can rewrite the operators (2) in the form of a Stieltjes integral as follows:

$$C_n(f;x) = \int_0^{b_n} f(t) \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt.$$

LEMMA 1 ([6]). For $C_n(t^s;x)$, $s = 0,1,2$, one has

$$\begin{aligned} C_n(1;x) &= 1, \\ C_n(t;x) &= x, \\ C_n(t^2;x) &= x^2 + \frac{x(b_n-x)}{n}. \end{aligned}$$

By direct calculation, we find the following equalities:

$$C_n((t-x)^2;x) = \frac{x(b_n-x)}{n}, \;\; C_n((t-x);x) = 0\,. \qquad (5)$$

LEMMA 2. For all $x \in (0,\infty)$, let $K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)$ be defined by (4), we have for $t < x$,

$$\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) = \int_0^t \frac{\partial}{\partial u} K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right) du \leq \frac{1}{(x-t)^2} \frac{x(b_n-x)}{n}. \qquad (6)$$

PROOF.

$$\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) = \int_0^t \frac{\partial}{\partial u}K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)du \le \int_0^t \left(\frac{x-u}{x-t}\right)^2 \frac{\partial}{\partial u}K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)du$$
$$= \frac{1}{(x-t)^2}\int_0^t (x-u)^2\frac{\partial}{\partial u}K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)du \le \frac{1}{(x-t)^2}C_n((u-x)^2;x).$$

From (5), it is easy to see that

$$\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) \le \frac{1}{(x-t)^2}\frac{x(b_n-x)}{n}.$$

REMARK. From Cauchy-Schwarz-Bunyakowsky inequality, one has

$$C_n(|t-x|\,;x) \le \left(C_n((t-x)^2;x)\right)^{\frac{1}{2}} = \sqrt{\frac{x(b_n-x)}{n}}. \quad (7)$$

3 Main Result

Now we can prove the main theorem. From (6), we can write the difference between $C_n(f;x)$ and $f(x)$ as a Lebesgue-Stieltjes integral,

$$C_n(f;x)-f(x) = \int_0^{b_n}[f(t)-f(x)]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \quad (8)$$

Since $f(t)\in DBV[0,b_n]$, we may rewrite (8) as follows:

$$\begin{aligned} & C_n(f;x)-f(x) \\ = & \int_0^x [f(t)-f(x)]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt + \int_x^{b_n}[f(t)-f(x)]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ = & -\int_0^x\left[\int_t^x f'(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt + \int_x^{b_n}\left[\int_x^t f'(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ = & -I_1(x)+I_2(x), \end{aligned}$$

where

$$I_1(x) = \int_0^x\left[\int_t^x f'(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \quad (9)$$

and

$$I_2(x) = \int_x^{b_n}\left[\int_x^t f'(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \quad (10)$$

For any $f(t) \in DBV[0, b_n]$, we decompose $f(t)$ into four parts as

$$\begin{aligned} f'(t) &= \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(t) + \frac{f'(x+) - f'(x-)}{2} sgn(t-x) \\ &\quad + \delta_x(t)\left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right)\right], \end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t \end{cases}.$$

If we use this equality in (9) and (10), we have the following expressions

$$\begin{aligned} I_1(x) &= \int_0^x \left\{ \int_t^x \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u)\left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right)\right] du \right\} \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \end{aligned}$$

and

$$\begin{aligned} I_2(x) &= \int_x^{b_n} \left\{ \int_x^t \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u)\left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right)\right] du \right\} \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt. \end{aligned}$$

Firstly, we evaluate $I_1(x)$:

$$\begin{aligned} I_1(x) &= \frac{f'(x+) + f'(x-)}{2} \int_0^x (x-t) \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \\ &\quad + \int_0^x \left[\int_t^x f'_x(u)\, du\right] \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \\ &\quad - \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-t) \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \\ &\quad + \left[f'(x) - \frac{f'(x+) + f'(x-)}{2}\right] \int_0^x \left[\int_t^x \delta_x(u)\, du\right] \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt. \end{aligned}$$

It is obvious that $\int\limits_t^x \delta_x(u)\,du = 0$. From this fact, we get

$$\begin{aligned} I_1(x) &= \frac{f'(x+)+f'(x-)}{2}\int\limits_0^x (x-t)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\int\limits_0^x\left[\int\limits_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &-\frac{f'(x+)-f'(x-)}{2}\int\limits_0^x (x-t)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \end{aligned} \tag{11}$$

Using a similar method, for evaluating $I_2(x)$, we find that

$$\begin{aligned} I_2(x) &= \frac{f'(x+)+f'(x-)}{2}\int\limits_x^{b_n} (t-x)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\int\limits_x^{b_n}\left[\int\limits_x^t f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &-\frac{f'(x+)-f'(x-)}{2}\int\limits_x^{b_n} (t-x)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned} -I_1(x)+I_2(x) &= \frac{f'(x+)+f'(x-)}{2}\int\limits_0^{b_n} (t-x)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\frac{f'(x+)-f'(x-)}{2}\int\limits_0^{b_n} |t-x|\,\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &-\int\limits_0^x\left[\int\limits_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\int\limits_x^{b_n}\left[\int\limits_x^t f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \end{aligned}$$

From the last expression, we can rewrite (8) as follows:

$$\begin{aligned} C_n(f;x)-f(x) &= \frac{f'(x+)+f'(x-)}{2}\int_0^{b_n}(t-x)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\frac{f'(x+)-f'(x-)}{2}\int_0^{b_n}|t-x|\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &-\int_0^x\left[\int_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt \\ &+\int_x^{b_n}\left[\int_x^t f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \end{aligned} \tag{13}$$

On the other hand, since

$$\int_0^{b_n}|t-x|\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt = C_n(|t-x|\,;x)$$

and

$$\int_0^{b_n}(t-x)\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt = C_n(t-x;x),$$

using these equalities in (13) and taking absolute value, we can express (13) as follows;

$$\begin{aligned} |C_n(f;x)-f(x)| &\le \left|\frac{f'(x+)+f'(x-)}{2}\right||C_n(t-x;x)| \\ &+\left|\frac{f'(x+)-f'(x-)}{2}\right||C_n(|t-x|\,;x)| \\ &+\left|-\int_0^x\left[\int_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt\right| \\ &+\left|\int_x^{b_n}\left[\int_x^t f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt\right|. \end{aligned} \tag{14}$$

According to (4), we write

$$\int_0^x\left[\int_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt = \int_0^x\left[\int_t^x f'_x(u)\,du\right]\frac{\partial}{\partial t}\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt. \tag{15}$$

Using partial integration on the right hand side of (15), we obtain

$$\int_0^x \left[\int_t^x f_x'(u)\,du\right] \frac{\partial}{\partial t}\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt = \int_0^x f_x'(t)\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt.$$

Thus

$$\left|-\int_0^x \left[\int_t^x f_x'(u)\,du\right] \frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt\right| \le \int_0^x |f_x'(t)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt$$

and

$$\begin{aligned}\left|-\int_0^x \left[\int_t^x f_x'(u)\,du\right] \frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)\right| &\le \int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt \\ &+ \int_{x-\frac{x}{\sqrt{n}}}^x |f_x'(t)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt.\end{aligned}$$

Since $f_x'(x) = 0$ and $\lambda_n(\frac{x}{b_n},\frac{t}{b_n}) \le 1$, one has

$$\int_{x-\frac{x}{\sqrt{n}}}^x |f_x'(t)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt = \int_{x-\frac{x}{\sqrt{n}}}^x |f_x'(t) - f_x'(x)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt \le \int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (f_x')\,dt.$$

Make the change of variables $t = x - \frac{x}{u}$, then

$$\int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (f_x')\,dt \le \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f_x') \int_{x-\frac{x}{\sqrt{n}}}^x dt.$$

From (6), we can write

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t)|\,\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt &\le \frac{x(b_n - x)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t)|\,\frac{dt}{(x-t)^2} \\ &= \frac{x(b_n - x)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t) - f_x'(x)|\,\frac{dt}{(x-t)^2} \\ &\le \frac{x(b_n - x)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (f_x')\frac{dt}{(x-t)^2}.\end{aligned}$$

Make the change of variables $t = x - \frac{x}{u}$ again, we have

$$\begin{aligned}\frac{x(b_n - x)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (f'_x) \frac{dt}{(x-t)^2} &= \frac{x(b_n - x)}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (f'_x) \frac{(\frac{x}{u^2})\, du}{(-\frac{x}{u})^2} \\ &= \frac{(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x)\end{aligned}$$

and hence, we obtain

$$\left| -\int_0^x [\int_t^x f'_x(u)\, du] \frac{\partial}{\partial t} K_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \le \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f'_x) + \frac{(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x).$$

Since

$$\frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f'_x) \le \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x),$$

it follows that

$$\begin{aligned}\frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f'_x) + \frac{(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x) &\le \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x) + \frac{2(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x) \\ &\le \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x).\end{aligned}$$

Therefore

$$\left| -\int_0^x \left[\int_t^x f'_x(u)\, du \right] \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \right| \le \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x). \tag{16}$$

Using a similar method for estimating, we have

$$\left| \int_x^{b_n} \left[\int_x^t f'_x(u)\, du \right] \frac{\partial}{\partial t} K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dt \right| \le \frac{b_n - x}{\sqrt{n}} \bigvee_x^{x+\frac{b_n - x}{\sqrt{n}}} (f'_x) + \frac{x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{b_n - x}{k}} (f'_x).$$

Furthermore, since

$$\frac{b_n - x}{\sqrt{n}} \bigvee_x^{x+\frac{b_n - x}{\sqrt{n}}} (f'_x) \le \frac{2(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{b_n - x}{k}} (f'_x),$$

we can write the following inequality

$$\begin{aligned}\frac{b_n-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{b_n-x}{\sqrt{n}}}(f'_x)+\frac{x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{b_n-x}{k}}(f'_x) &\leq \frac{2(b_n-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{b_n-x}{k}}(f'_x)\\ &\quad+\frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{b_n-x}{k}}(f'_x)\\ &\leq \frac{2b_n}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{b_n-x}{k}}(f'_x).\end{aligned}$$

Thus we get

$$\left|\int_x^{b_n}\left[\int_x^t f'_x(u)\,du\right]\frac{\partial}{\partial t}K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)dt\right|\leq\frac{2b_n}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{b_n-x}{k}}(f'_x). \tag{17}$$

Putting (5), (7), (16) and (17) in (14), we get (3), i.e.,

$$|C_n(f;x)-f(x)|\leq\left|\frac{f'(x+)-f'(x-)}{2}\right|\sqrt{\frac{x(b_n-x)}{n}}+\frac{2b_n}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{b_n-x}{k}}(f'_x).$$

This completes the proof of the theorem.

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