Rate Of Convergence Of Chlodowsky Operators For Functions With Derivatives Of Bounded Variation*

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Abstract

In the present paper, we estimate the rate of pointwise convergence of the Chlodowsky operators $C_n$ for functions, defined on the interval $[0, b_n]$ with derivatives of bounded variation, where $\lim_{n \to \infty} b_n = \infty$.

1 Introduction

For a function defined on the interval $[0, 1]$, the classical Bernstein operators are

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x), \quad 0 \leq x \leq 1, \quad n \geq 1,$$

where $p_{k,n}(x) = \left(\frac{n}{k}\right) x^k(1-x)^{n-k}$ is the Bernstein basis. Bernstein [1] used these operators (1) to give the first constructive proof of the Weierstrass theorem. It is well known that if we make the substitution $x = \frac{b_n}{b_{n+1}}$ and replacing the discrete values $f\left(\frac{k}{n}\right)$ by $f\left(\frac{k}{n}b_n\right)$, in the polynomial of Bernstein $B_n(f; x)$ corresponding to a function $f$ defined on $[0, 1]$, then one can obtain the following polynomials

$$C_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) \left(\frac{n}{k}\right) \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \quad 0 \leq x \leq b_n$$

(2)

where $(b_n)$ is a positive increasing sequence with the properties

$$\lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \frac{b_n}{n} = 0.$$

These operators are called Bernstein-Chlodowsky operators or Chlodowsky operators. Operators of type (2) were introduced by Chlodowsky [2] and further modified and studied by many authors [3-4]. Since the behaviour of Chlodowsky operators are very

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similar to the Bernstein operators, these operators allow us to investigate approximation properties of functions defined on the infinite interval $0 \leq x < \infty$ by using the similar techniques and methods on classical and modified Bernstein operators. For example, in [4] Ibikli and Karsli approximated integrable functions on the interval $[0, b_n]$ by what they called “Chlodowsky Type Durrmeyer operators” defined as follows: $D_n : BV[0, \infty) \to \mathcal{P}$,

$$D_n(f; x) = \frac{(n+1)}{b_n} \sum_{k=0}^{n} p_{k,n} \left( \frac{x}{b_n} \right) \int_{0}^{b_n} f(t) p_{k,n} \left( \frac{t}{b_n} \right) dt, \quad 0 \leq x \leq b_n,$$

where $p_{k,n}(x)$ is the Bernstein basis. They estimated the convergence rate of these $D_n$ operators for functions in $BV[0, \infty)$. After this study Karsli [5] obtained their rate of convergence for functions whose derivatives had bounded variation in $[0, \infty)$. It is useful to mention very recent papers by Karsli and Ibikli [6-7], which deal with the rate of pointwise convergence of the operators (2) and its Bézier variant in the space $BV[0, b_n]$ respectively.

The concern of this paper is to study the rate of convergence of operators $C_n$ to the limit $f$ of functions with derivatives of bounded variation on the interval $[0, b_n]$, $(n \to \infty)$ extending infinity. At the point $x$, which is a discontinuity of the first kind of the derivative, we shall prove that $C_n(f; x)$ converge to the limit $f(x)$.

Some authors studied some linear positive operators and obtained the rate of convergence for functions in $DBV(I)$. For example, Bojanic and Cheng investigated the rate of convergence of Hermite-Fejer polynomials for functions with derivatives of bounded variation [8] and they also investigated in the paper [9] the asymptotic behavior of Bernstein polynomials for functions in $DBV[0, 1]$ of all functions $f$ that can be written as

$$f(x) = f(0) + \int_{0}^{x} \Psi(t) dt, \quad x \in [0, 1],$$

where $\Psi \in BV[0, 1]$. We also mention some recent studies in this area by Gupta et al. [10], in which they estimated the rate of convergence of summation-integral-type operators for functions in $\gamma$-weighted space $DBV_\gamma(0, \infty), (\gamma \geq 0)$, and by Gupta et al. [11] and very recent papers by Karsli [12].

Let $DBV(I)$ denote the class of differentiable functions defined on a set $I \subset R$, whose derivatives are bounded variation on $I$,

$$DBV(I) = \{ f : f' \in BV(I) \}.$$

It is clear that the class of functions $DBV(I)$ considered here is much more general than the class of functions with continuous derivative on $I$.

For the sake of brevity, let the auxiliary function $f_x$ be defined by

$$f_x(t) = \begin{cases} 
  f(t) - f(x+), & x < t \leq b_n \\
  0, & t = x \\
  f(t) - f(x-), & 0 \leq t < x.
\end{cases}$$
The main theorem of this paper is as follows.

**THEOREM.** Let \( f \) be a function with derivatives of bounded variation on every finite subinterval of \([0, \infty)\) and \( \lim_{x \to \infty} f'(x) \) exists. Then for every \( x \in (0, \infty) \), we have

\[
|C_n(f; x) - f(x)| \leq \frac{|f'(x+) - f'(x-)|}{2} \sqrt{\frac{x(b_n - x)}{n}} + \frac{2b_n}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \frac{x + b_n - x - \frac{x}{4}}{x - \frac{x}{4}} (f'_x). \tag{3}
\]

where \( \int_a^b (f'_x) \) is the total variation of \( f'_x \) on \([a, b]\).

**2 Auxiliary Results**

In this section we give certain results, which are necessary to prove our main theorems.

As before we let

\[
\lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) := \int_0^t \frac{\partial}{\partial u} K_n \left( \frac{x}{b_n}, \frac{u}{b_n} \right) du
\]

where \( 0 \leq t \leq b_n \), and

\[
K_n \left( \frac{x}{b_n}, \frac{u}{b_n} \right) = \begin{cases} \sum_{kb_n \leq u} p_k, & 0 < u \leq b_n \\ 0, & u = 0 \end{cases}. \tag{4}
\]

Then \( \lambda_n (\frac{x}{b_n}, \frac{1}{b_n}) \leq 1 \). Since the operators (1) and (2) are special cases of Stieltjes integrals, alternatively we can rewrite the operators (2) in the form of a Stieltjes integral as follows:

\[
C_n(f; x) = \int_0^{b_n} f(t) \frac{\partial}{\partial u} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt.
\]

**LEMMA 1** ([6]). For \( C_n(t^s; x) \), \( s = 0, 1, 2 \), one has

\[
C_n(1; x) = 1,
\]

\[
C_n(t; x) = x,
\]

\[
C_n(t^2; x) = x^2 + \frac{x(b_n - x)}{n}.
\]

By direct calculation, we find the following equalities:

\[
C_n((t - x)^2; x) = \frac{x(b_n - x)}{n}, \quad C_n((t - x); x) = 0. \tag{5}
\]

**LEMMA 2.** For all \( x \in (0, \infty) \), let \( K_n \left( \frac{x}{b_n}, \frac{u}{b_n} \right) \) be defined by (4), we have for \( t < x \),

\[
\lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) = \int_0^t \frac{\partial}{\partial u} K_n \left( \frac{x}{b_n}, \frac{u}{b_n} \right) du \leq \frac{1}{(x-t)^2} \frac{x(b_n - x)}{n}. \tag{6}
\]
\section{Main Result}

Now we can prove the main theorem. From (6), we can write the difference between $C_n(f; x)$ and $f(x)$ as a Lebesgue-Stieltjes integral,

$$C_n(f; x) - f(x) = \int_0^b [f(t) - f(x)] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt.$$  \hspace{1cm} (8)

Since $f(t) \in DBV[0, b_n]$, we may rewrite (8) as follows:

$$C_n(f; x) - f(x) = \int_0^x [f(t) - f(x)] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt + \int_x^{b_n} [f(t) - f(x)] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt$$

$$= - \int_0^x \left[ \int_0^x f'(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt + \int_x^{b_n} \left[ \int_x^t f'(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt$$

$$= -I_1(x) + I_2(x),$$

where

$$I_1(x) = \int_0^x \left[ \int_0^x f'(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt$$  \hspace{1cm} (9)

and

$$I_2(x) = \int_x^{b_n} \left[ \int_x^t f'(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt.$$  \hspace{1cm} (10)
For any \( f(t) \in DBV[0, b_n] \), we decompose \( f(t) \) into four parts as

\[
f'(t) = \frac{1}{2} (f'(x+) + f'(x-)) + f'_x(t) + \frac{f'(x+) - f'(x-)}{2} \text{sgn}(t - x)
\]

\[+ \delta_x(t) \left[ f'(x) - \frac{1}{2} (f'(x+) + f'(x-)) \right],
\]

where

\[
\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}
\]

If we use this equality in (9) and (10), we have the following expressions

\[
I_1(x) = \int_0^x \left\{ \int_0^t \frac{1}{2} (f'(x+) + f'(x-)) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} \text{sgn}(u - x)
\]

\[+ \delta_x(u) \left[ f'(x) - \frac{1}{2} (f'(x+) + f'(x-)) \right] du \right\} \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt
\]

and

\[
I_2(x) = \int_0^{b_n} \left\{ \int_0^x \frac{1}{2} (f'(x+) + f'(x-)) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} \text{sgn}(u - x)
\]

\[+ \delta_x(u) \left[ f'(x) - \frac{1}{2} (f'(x+) + f'(x-)) \right] du \right\} \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt.
\]

Firstly, we evaluate \( I_1(x) \):

\[
I_1(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^x (x - t) \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt
\]

\[+ \int_0^x \left[ \int_t^x f'_x(u) du \right] \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt
\]

\[- \frac{f'(x+) - f'(x-)}{2} \int_0^x (x - t) \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt
\]

\[+ \left[ f'(x) - \frac{f'(x+) + f'(x-)}{2} \right] \int_0^x \left[ \int_t^x \delta_x(u) du \right] \frac{\partial}{\partial t} K_{b_n} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt.
\]
It is obvious that \( \int \delta_x(u) \, du = 0 \). From this fact, we get

\[
I_1(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^x (x-t) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt
\]

\[+ \int_0^x \left[ \int_t^x f'_x(u) \, du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt \]

\[- \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-t) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt. \tag{11} \]

Using a similar method, for evaluating \( I_2(x) \), we find that

\[
I_2(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^{b_n} (t-x) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt
\]

\[+ \int_x^{b_n} \left[ \int_x^t f'_x(u) \, du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt \]

\[- \frac{f'(x+) - f'(x-)}{2} \int_0^{b_n} (t-x) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt. \tag{12} \]

Combining (11) and (12), we get

\[-I_1(x) + I_2(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^{b_n} (t-x) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt
\]

\[+ \frac{f'(x+) - f'(x-)}{2} \int_0^{b_n} |t-x| \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt \]

\[- \int_0^x \left[ \int_t^x f'_x(u) \, du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt
\]

\[+ \int_x^{b_n} \left[ \int_x^t f'_x(u) \, du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt. \]
From the last expression, we can rewrite (8) as follows:

\[ C_n(f; x) - f(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^{b_n} (t - x) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt + \frac{f'(x+) - f'(x-)}{2} \int_0^{b_n} |t - x| \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt + \frac{f'(x+)}{2} \int_0^{b_n} \left[ \int_0^{x} f'_x(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt - \frac{f'(x-)}{2} \int_0^{b_n} \left[ \int_x^{b_n} f'_x(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt. \] (13)

On the other hand, since

\[ \int_0^{b_n} \left| t - x \right| \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt = C_n(\left| t - x \right|; x) \]

and

\[ \int_0^{b_n} (t - x) \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt = C_n(t - x; x) \]

using these equalities in (13) and taking absolute value, we can express (13) as follows;

\[ |C_n(f; x) - f(x)| \leq \left| \frac{f'(x+) + f'(x-)}{2} \right| |C_n(t - x; x)| + \left| \frac{f'(x+) - f'(x-)}{2} \right| |C_n(\left| t - x \right|; x)| + \left| \frac{f'(x+)}{2} \right| \left[ \int_0^{b_n} \left[ \int_0^{x} f'_x(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt \right] - \left| \frac{f'(x-)}{2} \right| \left[ \int_0^{b_n} \left[ \int_x^{b_n} f'_x(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt \right]. \] (14)

According to (4), we write

\[ \int_0^{x} \left[ \int_0^{x} f'_x(u) du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt = \int_0^{x} \left[ \int_0^{x} f'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt. \] (15)
Using partial integration on the right hand side of (15), we obtain
\[
\int_0^x \left[ \int_t^x f'_x(u) \,du \right] \frac{\partial}{\partial t} \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt = \int_0^x f'_x(t) \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt.
\]

Thus
\[
\left| - \int_0^x \left[ \int_t^x f'_x(u) \,du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt \right| \leq \int_0^x |f'_x(t)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt
\]

and
\[
\left| - \int_0^x \left[ \int_t^x f'_x(u) \,du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \leq \int_0^x |f'_x(t)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt
\]

\[
+ \int_0^x |f'_x(t)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt.
\]

Since \( f'_x(x) = 0 \) and \( \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq 1 \), one has
\[
\int_{x - \sqrt{n}}^x |f'_x(t)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt = \int_{x - \sqrt{n}}^x |f'_x(t) - f'_x(x)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt \leq \int_{x - \sqrt{n}}^x \sqrt{|f'_x|} \,dt.
\]

Make the change of variables \( t = x - \frac{u}{n} \), then
\[
\int_{x - \sqrt{n}}^x \sqrt{|f'_x|} \,dt \leq \sqrt{x - \frac{1}{n}} \int_{x - \frac{1}{n}}^x dt.
\]

From (6), we can write
\[
\int_{x - \sqrt{n}}^x |f'_x(t)| \lambda_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \,dt \leq \frac{x(b_n - x)}{n} \int_{0}^{x - \sqrt{n}} |f'_x(t)| \frac{dt}{(x - t)^2}
\]

\[
= \frac{x(b_n - x)}{n} \int_{0}^{x - \sqrt{n}} |f'_x(t) - f'_x(x)| \frac{dt}{(x - t)^2}
\]

\[
\leq \frac{x(b_n - x)}{n} \int_{0}^{x - \sqrt{n}} \sqrt{|f'_x|} \frac{dt}{(x - t)^2}.
\]
Make the change of variables \( t = x - \frac{x}{u} \) again, we have
\[
\frac{x(b_n - x)}{n} \left[ \int_{0}^{x} \frac{f'(x)'}{(x-t)^2} \right] = \frac{x(b_n - x)}{n} \int_{1}^{x} f'(x) \left( \frac{x}{u} \right) \left( \frac{x}{u} \right) du
\]
and hence, we obtain
\[
\left| - \int_{0}^{x} \int_{t}^{x} f'(u) du \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \leq \frac{x}{\sqrt{n}} \int_{x - \frac{x}{\sqrt{n}}}^{x} (f'_x) + \frac{[\sqrt{n}] x}{n} \sum_{k=1}^{x} (f'_x).
\]
Since
\[
\frac{x}{\sqrt{n}} \int_{x - \frac{x}{\sqrt{n}}}^{x} (f'_x) \leq \frac{2x}{n} \sum_{k=1}^{x} (f'_x),
\]
it follows that
\[
\frac{x}{\sqrt{n}} \int_{x - \frac{x}{\sqrt{n}}}^{x} (f'_x) + \frac{[\sqrt{n}] x}{n} \sum_{k=1}^{x} (f'_x) \quad \leq \quad \frac{2x}{n} \sum_{k=1}^{x} (f'_x) + \frac{2(b_n - x)}{n} \sum_{k=1}^{x} (f'_x)
\]
\[
\leq \frac{2b_n}{n} \sum_{k=1}^{x} (f'_x).
\]
Therefore
\[
\left| - \int_{0}^{x} \int_{t}^{x} f'(u) du \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \leq \frac{2b_n}{n} \sum_{k=1}^{x} (f'_x). \quad (16)
\]
Using a similar method for estimating, we have
\[
\left| \int_{x}^{b_n} \int_{x}^{t} f'(u) du \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) dt \right| \leq \frac{b_n - x}{\sqrt{n}} \int_{x}^{x + \frac{b_n - x}{\sqrt{n}}} (f'_x) + \frac{[\sqrt{n}] x + \frac{b_n - x}{\sqrt{n}}}{n} \sum_{k=1}^{x} (f'_x).
\]
Furthermore, since
\[
\frac{b_n - x}{\sqrt{n}} \int_{x}^{x + \frac{b_n - x}{\sqrt{n}}} (f'_x) \leq \frac{2(b_n - x)}{n} \sum_{k=1}^{x} (f'_x),
\]
we can write the following inequality

\[
\frac{b_n - x}{\sqrt{n}} \left( x + \frac{b_n - x}{2} \right) + \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x) \leq \frac{2(b_n - x)}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x) + \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x)
\]

\[
\leq \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x).
\]

Thus we get

\[
\left| \int_x^t \left[ \int_x^u f'_x(u) \, du \right] \frac{\partial}{\partial t} K_n \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \, dt \right| \leq \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x). \tag{17}
\]

Putting (5), (7), (16) and (17) in (14), we get (3), i.e.,

\[
|C_n(f; x) - f(x)| \leq \frac{f'(x^+) - f'(x^-)}{2} \sqrt{x(b_n - x)} + \frac{2b_n}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{x + b_n - x} (f'_x).
\]

This completes the proof of the theorem.

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**References**


