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Optimal Control Of Production Inventory Systems With Deteriorating Items And Dynamic Costs*

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Abstract

The objective of the present research is to make use of optimal control theory to solve a production inventory problem with deteriorating items and dynamic costs.

1 Introduction

A great deal of effort has been focused on the modelling of the production planning problem in a deterministic environment. Early static models, such as the classical economic order quantity (EOQ) model, assume a constant demand rate and are solved using classical optimization tools. The dynamic counterparts, also known as generalized economic order quantity (GEOQ) models, assume a time varying demand rate. They are tackled using either classical optimization or optimal control theory. Which of the two theories is more appropriate to solve a particular problem and what are the possibilities and limitations of each of them are questions that are outside the scope of this paper. For the interested reader see, for example, Axsäter [3] who provides such a critical evaluation.

In this paper we are using optimal control theory, which is a branch of mathematics particularly well suited to find optimal ways to control a dynamic system. It proved its efficiency in many areas of operations research such as finance [7, 9, 23], economics [2, 10, 16], marketing [11, 22], maintenance [18, 19], environment and transportation [4, 17], and the consumption of natural resources [1, 8].

We are especially interested in the application of optimal control theory to the production planning problem. During the last two decades, various authors attacked this research direction. We state here some of them: Bounkhel and Tadj [5], Hedjar *et al.* [14], Khemlnitsky and Gerchak [17], Riddals and Bennett [20], Salama [21], and Zhang *et al.* [25].

The problem of interest to us is related to one of the assumptions of the classical EOQ model and has received very little attention in the literature. It concerns the various unit costs involved, which are assumed to be known and constant. In a GEOQ model, the demand rate, instead of being uniform, is assumed to be dynamic, which is

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a more realistic assumption. It seems natural to further generalize the GEOQ model by relaxing the assumption of constant unit costs. Indeed, except for the ordering cost which is paid only once per cycle, costs can be affected by time in various ways. For example, the unit cost of an item may decrease with time as new products are produced. Also, the holding cost may change as time goes by.

Only very few papers that address this issue of dynamic costs came to our attention. The first paper seems to be that of Giri *et al.* [12], who compute the optimal policy of an EOQ model with dynamic costs. The model they proposed is very basic though, since they consider the very special case where the holding and ordering costs are linear functions of time. The other shortcomings of that paper is that the items deterioration rate is also a linear function of time, and the algorithm they proposed in order to solve the problem is only valid as long as the demand rate is a linear function of time. Teng *et al.* [24], in a more recent paper, assume that only one of the unit costs, namely the purchasing or production cost, is time-varying.

The goal of this paper is to generalize the models of these papers in various ways. First, the costs in our model are general functions of time instead of being linear functions. Second, the on-hand inventory deteriorates and the deterioration rate is a general function of time instead of being a linear function. Importance of items deteriorating in inventory modelling in now widely acknowledged, as shown by the recent survey of Goyal and Giri [13]. Third, the demand rate is a general function of time. Finally, an optimal control approach is used instead of an optimization approach. We aim primarily at deriving the necessary and sufficient conditions for the optimal solution of the model under consideration and from there introduce illustrative examples with their numerical verifications which explain the applications of the theoretical results to some real life problems. To construct the objective function of the problem, we assume that the firm has set an inventory goal level, a production goal rate, and a deterioration goal rate and penalties are incurred when the inventory level, the production rate, and the deterioration rate deviate from their respective goals.

The paper deals with both continuous and periodic-review policies. We solve the first model by applying Pontriyagin maximum principle. In the second model, the periodic-review model, we use Lagrange multiplier method to minimize an objective function subject to some difference equation. In both models we derive explicit optimal policies that can be used by managers to augment their capabilities in the decision-making process.

The rest of the paper is organized as follows. Following this introduction, all the notation needed in the sequel is stated in Section 2. In Section 3, we build and solve the continuous-review model. Our main result in this section is stated in Theorem 3.1. An illustrative example of this result is given. We do similarly for the periodic-review model in Section 4 and the main result in this section is stated in Theorem 4.1. The last section concludes the paper.

2 Notation

Let us consider a manufacturing firm producing a single product. We first introduce the notation that is independent of the time: T : length of the planning horizon, ρ :

constant nonnegative discount rate, I_0 : initial inventory level. Next, the notation that depends on time, and that we divide into two categories. The monetary parameters which are, as we mentioned in the Introduction, dynamic: $h(t)$: holding penalty cost rate at time t, $K(t)$: production penalty cost rate at time t. The nonmonetary parameters that depend on time may also be divided into two groups. The first group comprises the state variable, the control variable, and the exogenous functions: $I(t)$: inventory level at time t, $P(t)$: production rate at time t, $D(t)$: demand rate at time t, $\theta(t)$: deterioration rate at time t. The second group comprises the goals: $\hat{I}(t)$: inventory goal level at time t, $\hat{P}(t)$: production goal rate at time t. The interpretation of the goal rates are as follows: The inventory goal level $\hat{I}$ is a safety stock that the company wants to keep on hand. For example, $\hat{I}$ could be 200 units of the finished product during $[0, t_1]$ and 180 units during $[t_1, T]$. The production goal rate $\hat{P}$ is the most efficient rate desired by the firm.

All functions are assumed to be non-negative, continuous and differentiable functions. As we mentioned in the Introduction, we will be considering both cases where the firms adopts a continuous-review and a periodic-review policy. The notation that does not depend on time will be the same for both cases. Concerning the notation that depends on time, the variable t will be used in the continuous time case and the variable k will be used in the discrete time case.

3 Model Solution

3.1 Continuous-Review Policy

We first assume that the firm adopts a continuous-review policy. The dynamics of the inventory level $I(t)$ are governed by the following differential equation:

$$\frac{d}{dt}I(t) = P(t) - D(t) - \theta(t)I(t), \tag{1}$$

with $I(0) = I_0$. The model is presented as an optimal control problem with one state variable (inventory level) and one control variable (rate of manufacturing). The problem $(\mathcal{P}_{cr})$ associated to this model is to minimize the following objective function

$$\min_{P(t)\geq 0} J(P, I) = \int_0^T F(t, I(t), P(t))dt,$$

subject to the state equation (1) where

$$F(t, I(t), P(t)) = \frac{1}{2}e^{-\rho t}\left\{h(t)\Delta^2 I(t) + K(t)\Delta^2 P(t)\right\}, \tag{2}$$

and

$$\Delta I(t) = I(t) - \hat{I}(t), \quad \Delta P(t) = P(t) - \hat{P}(t).$$

We adopt the widely used quadratic objective function of Holt, Modigliani, Muth, and Simon (HMMS) [15]. The interpretation of this objective function is that penalties are incurred when the inventory level and production rate deviate from their respective goals. Note that we are determining the present value of future costs by discounting them using the appropriate cost of capital. This is necessary because cash flows in different time periods cannot be directly compared since most people prefer money sooner rather than later. This is a simple interpretation of the fact that a dollar in your hand today is worth more than a dollar you may receive at some point in the future. The following is the main result of this section.

THEOREM 1. Necessary conditions for the pair (P, I) to be an optimal solution of problem $(\mathcal{P}_{cr})$ are

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\Delta I(t) + \left[\frac{\frac{d}{dt}K(t)}{K(t)} - \rho\right]\frac{d}{dt}\Delta I(t) \\ &\quad + \left[\frac{d}{dt}\theta(t) - \theta(t)\left[\left(\frac{\frac{d}{dt}K(t)}{K(t)} - \rho\right) - \theta(t)\right] - \frac{h(t)}{K(t)}\right]\Delta I(t) \end{aligned} \tag{3}$$

and

$$I(0) = I_0, \quad P(T) = \hat{P}(T), \quad P(t) \geq 0, \ \forall t \in [0, T].$$

PROOF. We will use Pontryagin's maximum principle to solve the above problem $(\mathcal{P}_{cr})$. The Hamiltonian function is

$$H(t, I(t), P(t), \lambda(t)) = -F(t, I(t), P(t)) + \lambda(t) f(t, I(t), P(t)), \tag{4}$$

where $f(t, I(t), P(t))$ is the right-hand side of the state equation (1) and λ is the adjoint function associated with this constraint. Assume (P, I) is an optimal solution to problem $(\mathcal{P}_{cr})$, then

$$H(t, I(t), P(t), \lambda(t)) \geq H(t, I(t), \tilde{P}(t), \lambda(t)), \quad \text{for all } \tilde{P}(t) \geq 0, \tag{5}$$

$$-\frac{d}{dt}\lambda(t) = \frac{\partial}{\partial I}H(t, I(t), P(t), \lambda(t)), \tag{6}$$

$$I(0) = I_0, \qquad \lambda(T) = 0. \tag{7}$$

Equation (5) is equivalent to

$$\frac{\partial}{\partial P}H(t, I(t), P(t), \lambda(t)) = 0,$$

which is equivalent to

$$\lambda(t) = K(t)e^{-\rho t}\Delta P(t). \tag{8}$$

Equation (6) is equivalent to

$$\frac{d}{dt}\lambda(t) = h(t)e^{-\rho t}\Delta I(t) + \lambda(t)\theta(t). \tag{9}$$

Now, combining Equation (8) and Equation (9) yields

$$\frac{d}{dt}\Delta P(t) + \left[\frac{\frac{d}{dt}K(t)}{K(t)} - \rho - \theta(t)\right]\Delta P(t) = \frac{h(t)}{K(t)}\Delta I(t). \tag{10}$$

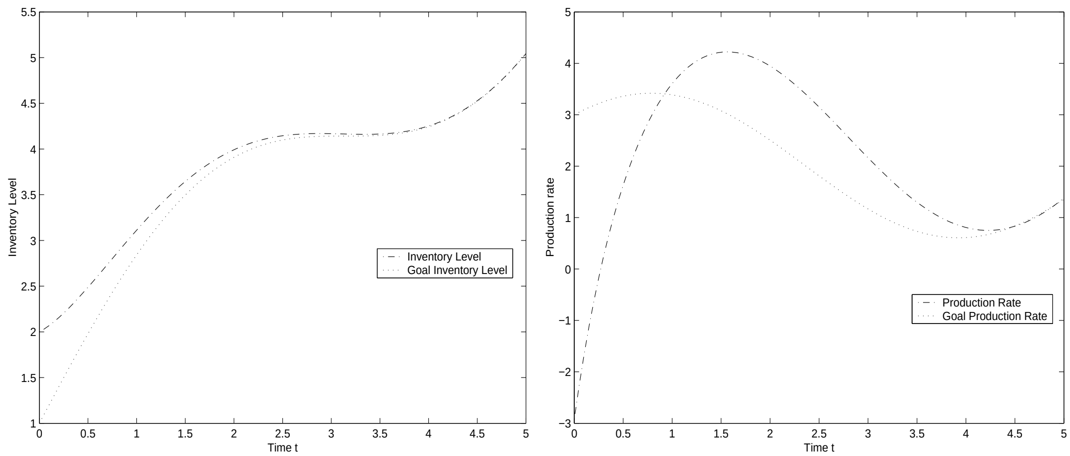

Figure 1: Optimal solution for Example 3.1.

Using the fact that all the goal rates must satisfy the state equation we get

$$\frac{d}{dt}\hat{I}(t) = \hat{P}(t) - D(t) - \theta(t)\hat{I}(t), \tag{11}$$

which yields with the state equation (1)

$$\Delta P(t) = \frac{d}{dt}\Delta I(t) + \theta(t)\Delta I(t). \tag{12}$$

Combining this equation with (10) yields the following second order differential equation (3). The fact that $P(T) = \hat{P}(T)$ follows directly from the equations (8) and (7). Thus the proof is complete.

REMARK 1. The problem studied in the present paper is convex and its convexity is ensured by the convexity of the function $I \mapsto F(t, I(t), P(t))$ (for other convex and nonconvex problems see for instance [5]).

EXAMPLE 1. We present an illustrative example of the obtained results. Take $T = 5$, $\rho = 0.001$, $I_0 = 2$, $h(t) = K(t) = 1+t$, $D(t) = 2\sin(t)+3$, $\theta(t) = 0.001+0.001t$, $\hat{I}(t) = 1+t+\sin(t)$, and $\hat{P}(t)$ is computed from the state equation. Figure 1 shows the variations of the optimal inventory level (left) and the optimal production rate (right).

3.2 Periodic-Review Policy

We now assume that the firm adopts a periodic-review policy. The dynamics of the inventory level $I(k)$ are governed by the following difference equation:

$$I(k+1) - I(k) = P(k) - D(k) - \theta(k)I(k), \quad 1 \le k \le N, \tag{1}$$

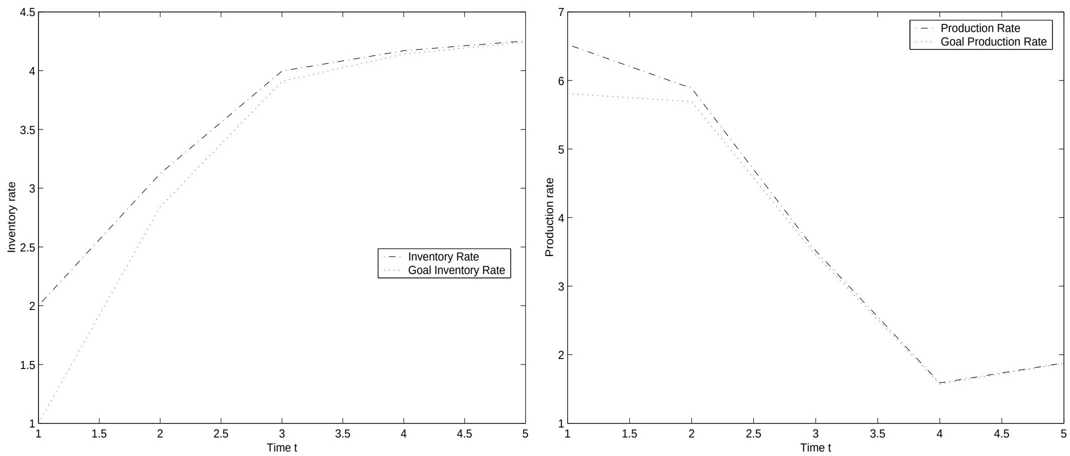

Figure 2: Optimal solution for Example 4.1.

where N is the length of the planning horizon. Introduce the shifted variables $\Delta I(k)$ and $\Delta P(k)$ as $\Delta I(k) = I(k) - \hat{I}(k)$ and $\Delta P(k) = P(k) - \hat{P}(k)$. Note that the goal rates $\hat{I}(k)$ and $\hat{P}(k)$ must satisfy the equation

$$\hat{I}(k+1) - \hat{I}(k) = \hat{P}(k) - D(k) - \theta(k)\hat{I}(k), \quad 1 \leq k \leq N, \tag{2}$$

and therefore equations (1) and (2) lead to

$$\Delta I(k+1) = \Delta P(k) + (1 - \theta(k))\Delta I(k). \tag{3}$$

The problem $(\mathcal{P}_{pr})$ associated to this model is to minimize the following objective function

$$\min_{P \geq 0} J(P, I) = \sum_{k=1}^{N} F(k, I(k), P(k)),$$

subject to the state equation (3) where

$$F(k, I(k), P(k)) = \frac{1}{2} \frac{1}{(1+\rho)^{k-1}} \left\{ h(k)\Delta^2 I(k) + K(k)\Delta^2 P(k) \right\}. \tag{4}$$

The following is the main result of this section.

THEOREM 2. A necessary condition for the pair $(P, I) = (P(k), I(k))_{1 \leq k \leq N}$ to be an optimal solution of problem $(\mathcal{P}_{pr})$ is

$$\left\{ \begin{array}{lcl} I(k+1) & = & \hat{I}(k+1) + [1 - \theta(k)]\,[1 - \alpha(k+1)]\,, \\ \Delta P(k) & = & -\alpha(k+1)\,[1 - \theta(k)]\,\Delta I(k), \end{array} \right. \tag{5}$$

where $\alpha(k+1) = \frac{(1+\rho)^{k-1}s(k+1)}{K(k)+(1+\rho)^{k-1}s(k+1)}$, and $s(k)$ is given in the proof.

PROOF. In order to solve our periodic-review problem, we introduce the Lagrangian function

$$L(I,P,\lambda) = \sum_{k=1}^{N} \left[F(k,I(k),P(k)) + \lambda(k+1)f(k,I(k),P(k))\right], \tag{6}$$

where $f(k,I(k),P(k)) = -\Delta I(k+1) + \Delta P(k) + (1-\theta(k))\Delta I(k)$ and $\lambda(k+1)$ is the Lagrange multiplier associated with the difference equation constraint (3). Thus, the necessary optimality conditions for (P,I) to be an optimal solution for $(\mathcal{P}_{pr})$ are

$$\frac{\partial L}{\partial \Delta P(k)} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \Delta I(k)} = 0 \quad (1 \le k \le N).$$

After some computations, these equations yield respectively

$$\Delta P(k) = -(1+\rho)^{k-1}K(k)^{-1}\lambda(k+1) \text{ and } \lambda(k) = \frac{h(k)}{(1+\rho)^{k-1}}\Delta I(k) + \left[1-\theta(k)\right]\lambda(k+1) \tag{7}$$

The sweep method (by Bryson and Ho [6]) assumes that $\lambda(k) = s(k)\Delta I(k)$, with $s(k) > 0$, for $k = 0,\ldots,N$. Subtituting the last equation and (3) into (7) yields

$$\Delta P(k) = -\frac{(1+\rho)^{k-1}s(k+1)\left[1-\theta(k)\right]}{K(k)+(1+\rho)^{k-1}s(k+1)}\Delta I(k) \quad \text{and} \tag{8}$$

$$s(k) = s(k+1)\left[1-\theta(k)\right]^2\left[1 - \frac{(1+\rho)^{k-1}s(k+1)}{K(k)+(1+\rho)^{k-1}s(k+1)}\right] + \frac{h(k)}{(1+\rho)^{k-1}}. \tag{9}$$

By the fact that $\Delta P(N) = 0$ and by using the equations (8) and (9) we get $s(N) = \frac{h(N)}{(1+\rho)^{N-1}}$. The recursive equation (9) for $s(k)$ is solved backwards starting from the end point N. Therefore, the desired optimal solutions are given by (5). Thus the proof is complete.

REMARK 2. Note that it is not hard to show that the necessary optimality condition (5) is also sufficient by the convexity of the function F as a function of I (see for instance [5]).

EXAMPLE 2. Taking the same data as in Example 1., Figure 2 shows the variations of the optimal inventory level (left) and the optimal production rate (right).

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