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A Superquadratic Method For Solving Variational Inclusions Under Weak Conditions*

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Abstract

For solving variational inclusions of the form $0 \in f(x) + F(x)$, in [13, 14], the authors proved the convergence of the following method inspired by Hummel-Seebeck $0 \in f(x_k) + \frac{1}{2}(\nabla f(x_k) + \nabla f(x_{k+1}))(x_{k+1} - x_k) + F(x_{k+1})$ where f is a function whose second Fréchet derivative $\nabla^2 f$ satisfies a Lipschitz condition or a Hölder condition. In this paper, we extend these results by assuming a center-Hölder condition on $\nabla^2 f$.

1 Introduction

This paper is concerned with the problem of approximating a solution of variational inclusions of the form

$$0 \in f(x) + F(x) \tag{1}$$

where f is a function and F is a set-valued map defined in two Banach spaces X and Y. This kind of inclusion is an abstract model for various problems : variational problems, optimization and control theory, operations research, complementarity problems, mathematical programming and engineering sciences [10, 15, 16]. For solving (1), the following method has been introduced in [13],

$$0 \in f(x_k) + \frac{1}{2}\Big(\nabla f(x_k) + \nabla f(x_{k+1})\Big)(x_{k+1} - x_k) + F(x_{k+1}), \tag{2}$$

where f is a function such that its second Fréchet derivative $\nabla^2 f$ satisfies a Lipschitz condition. The existence of a sequence (x_k) defined by (2) and its convergence to a solution x^* of (1) has been also proved.

Following this work, in [14], the authors extended these results by applying a Hölder condition on the second Fréchet derivative $\nabla^2 f$. This condition reads as follows:

$$\exists\ K > 0,\ \alpha \in (0, 1],\ \text{such that}\ \|\nabla^2 f(x) - \nabla^2 f(y)\| \le K\|x - y\|^{\alpha}, \quad \forall\ x, y \in \Omega,$$

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where Ω is a neighborhood of x^*. We can notice that when $\alpha = 1$, we have the Lipschitz condition for $\nabla^2 f$.

In this study, we are interested in the convergence of the method (2) when $\nabla^2 f$ satisfies a center-Hölder assumption :

$$\exists \alpha_0 \in (0,1], \text{ such that} \, \forall \; x \in \Omega, \; \|\nabla^2 f(x) - \nabla^2 f(x^*)\| \leq K_0 \|x - x^*\|^{\alpha_0}.$$

The inspiration for considering such a condition comes from [1, 2]. Let us remark that, in some cases, the center-Hölder condition holds whereas the Hölder condition doesn't. Thus, this condition of center-Hölder is weaker than the Hölder one hence allows us to refine the result established in [13, 14].

Throughout, we denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r and by $\|.\|$ all the norms. The distance from a point $x \in X$ and a subset $A \subset X$ is defined as $dist(x, A) = \inf_{y \in A}\{\|x - y\|\}$.

Recall that a set-valued $\Gamma : X \longrightarrow 2^Y$ is said to be M-pseudo-Lipschitz around $(x_0, y_0) \in \operatorname{graph} \Gamma$ if there exist constants a and b such that

$$e(\Gamma(x_1) \cap \mathbb{B}_a(y_0), \Gamma(x_2)) \leq M\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{B}_b(x_0),$$

where the excess e from the set A to the set C is defined by $e(C, A) = \sup_{x \in C} \; dist(x, A)$.

The pseudo-Lipschitz property has been introduced by J.-P. Aubin and one refers to it as Aubin-continuity [3, 4, 18]. This property is equivalent to the metric regularity and to linear openness, for more details, the reader could refer to [7, 8, 9]. This concept is necessary for our study and often used for solving inclusions of the form (1), see [5, 11, 17].

2 Convergence analysis

The main result is the following theorem:

THEOREM 1. Let x^* be a solution of (1) and let f be a function whose second Fréchet derivative $\nabla^2 f$ satisfies a center-Hölder condition with a constant K_0 and exponent α_0 on a neighborhood Ω of x^*. If the set-valued map $(f+F)^{-1}$ is M-pseudo-Lipschitz around $(0, x^*)$ then for every $c > \dfrac{MK_0(2\alpha_0^2 + 9\alpha_0 + 8)}{2(\alpha_0+1)(\alpha_0+2)}$, one can find $\delta > 0$ such that for every starting point $x_0 \in \mathbb{B}_\delta(x^*)$, there exists a sequence (x_k) for (1), defined by (2), which satisfies

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^{\alpha_0 + 2} \tag{3}$$

that is, (x_k) is superquadratically convergent to x^*.

In the proof of Theorem 1, we need two lemmas:

LEMMA 1. If $f : X \to Y$ is a function such that ∇f is Lipschitz then the following are equivalent:

(i) The mapping $(f+F)^{-1}$ is pseudo-Lipschitz around (y^*, x^*).
(ii) The mapping $[f(x^*) + \frac{1}{2}(\nabla f(x^*) + \nabla f(\cdot))(\cdot - x^*) + F(\cdot)]^{-1}$ is pseudo-Lipschitz around (y^*, x^*).

The reader can consult the proof of this lemma in [13].

LEMMA 2. Let (X,ρ) be a complete metric space, let ϕ be a map from X into the closed subsets of X, let $\eta_0 \in X$ and let r and λ be such that $0 \le \lambda < 1$ and
(a) dist $(\eta_0, \phi(\eta_0)) \le r(1-\lambda)$,
(b) $e(\phi(x_1) \cap \mathbb{B}_r(\eta_0), \phi(x_2)) \le \lambda\ \rho(x_1,x_2),\ \forall x_1, x_2 \in \mathbb{B}_r(\eta_0)$,
then ϕ has a fixed point in $\mathbb{B}_r(\eta_0)$. That is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $\mathbb{B}_r(\eta_0)$.

This lemma is a generalization of a fixed-point theorem of Ioffe-Tikhomirov [12]. The reader can consult its proof in [6].

For a better understanding of Theorem 1, let us introduce a few notation. For $k \in \mathbb{N}$ and $x_k \in X$, we define the maps $P : X \to 2^Y$ and $\phi_k : X \to 2^X$ by

$$P(x) = f(x^*) + \frac{1}{2}\Big(\nabla f(x^*) + \nabla f(x)\Big)(x - x^*) + F(x) \quad \text{and} \quad \phi_k(x) = P^{-1}[Z_k(x)],$$

where

$$Z_k(x) = f(x^*) + \frac{1}{2}\Big(\nabla f(x^*) + \nabla f(x)\Big)(x - x^*) - f(x_k) - \frac{1}{2}\Big(\nabla f(x_k) + \nabla f(x)\Big)(x - x_k).$$

We remark that x_1 is a fixed point of ϕ_0 if and only if if we have

$$0 \in f(x_0) + \frac{1}{2}\Big(\nabla f(x_0) + \nabla f(x_1)\Big)(x_1 - x_0) + F(x_1).$$

Proceeding by induction, we show that the function ϕ_k has a fixed point x_{k+1} in X. Thus, we have the existence of a sequence (x_k) defined by (2) which satisfies (3).

PROOF. The map $(f+F)^{-1}$ is M-pseudo-Lipschitz around $(0, x^*)$ then there exist positive numbers a and b such that

$$e(P^{-1}(y') \cap \mathbb{B}_a(x^*), P^{-1}(y'')) \le M\|y' - y''\|,\ \forall y', y'' \in \mathbb{B}_b(0). \tag{4}$$

Choose $\delta > 0$ such that

$$\delta < \min\left\{a, \frac{1}{\sqrt[\alpha_0+1]{c}}, \left[\frac{2b(\alpha_0+1)(\alpha_0+2)}{K_0(2\alpha_0^2 + 9\alpha_0 + 8)}\right]^{\frac{1}{\alpha_0+2}}, \left[\frac{2b(\alpha_0+1)(\alpha_0+2)}{K_0(18\alpha_0^2 + 57\alpha_0 + 40)}\right]^{\frac{1}{\alpha_0+2}}\right\}. \tag{5}$$

We apply Lemma 2 to the map ϕ_0 with $\eta_0 = x^*$ and r and λ are numbers to be set. Let us check that assertions (a) and (b) of this lemma are satisfied.

From the definition of the excess e, we have

$$\text{dist } (x^*, \phi_0(x^*)) \le e(P^{-1}(0) \cap \mathbb{B}_\delta(x^*), \phi_0(x^*)). \tag{6}$$

For all $x_0 \in \mathbb{B}_\delta(x^*)$ such that $x_0 \neq x^*$, we have

$$
\begin{aligned}
& \|Z_0(x^*)\| \\
= \ & \|f(x^*) - f(x_0) - \frac{1}{2}\Big(\nabla f(x_0) + \nabla f(x^*)\Big)(x^* - x_0)\| \\
= \ & \|f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x^* - x_0)^2 \\
& -\frac{1}{2}(\nabla f(x^*) - \nabla f(x_0) - \nabla^2 f(x_0)(x^* - x_0))(x^* - x_0)\| \\
\leq \ & \| \int_0^1 (1-t)\nabla^2 f(x_0 + t(x^* - x_0))(x^* - x_0)^2 dt - \frac{1}{2}\nabla^2 f(x_0)(x^* - x_0)^2\| \\
& +\frac{1}{2}\|x^* - x_0\|.\| \int_0^1 \nabla^2 f(tx^* + (1-t)x_0)dt(x^* - x_0) - \int_0^1 \nabla^2 f(x_0)dt(x^* - x_0)\| \\
\leq \ & \| \int_0^1 (1-t)\Big[\nabla^2 f(x_0 + t(x^* - x_0)) - \nabla^2 f(x_0)\Big](x^* - x_0)^2 dt\| \\
& +\frac{1}{2}\|x^* - x_0\|^2 \int_0^1 \|\nabla^2 f(tx^* + (1-t)x_0) - \nabla^2 f(x_0)\|dt \\
\leq \ & \| \int_0^1 (1-t)\Big[\nabla^2 f(x_0 + t(x^* - x_0)) - \nabla^2 f(x^*) + \nabla^2 f(x^*) - \nabla^2 f(x_0)\Big] \\
& (x^* - x_0)^2 dt\| + \frac{1}{2}\|x^* - x_0\|^2 \int_0^1 \|\nabla^2 f(tx^* + (1-t)x_0) - \nabla^2 f(x^*)dt \\
& +\nabla^2 f(x^*) - \nabla^2 f(x_0)\| \\
\leq \ & K_0\|x^* - x_0\|^2 \int_0^1 (1-t)\Big[\|x_0 + t(x^* - x_0) - x^*\|^{\alpha_0} + \|x^* - x_0\|^{\alpha_0}\Big]dt \\
& +\frac{1}{2}K_0\|x^* - x_0\|^2 \int_0^1 \Big[\|(1-t)(x_0 - x^*)\|^{\alpha_0} + \|x^* - x_0\|^{\alpha_0}\Big]dt \\
\leq \ & K_0\|x^* - x_0\|^{\alpha_0+2} \int_0^1 \Big[(1-t)^{\alpha_0+1} + 1 - t\Big]dt \\
& +\frac{1}{2}K_0\|x^* - x_0\|^{\alpha_0+2} \int_0^1 \Big[(1-t)^{\alpha_0} + 1\Big]dt \\
\leq \ & \frac{K_0(\alpha_0+4)}{2(\alpha_0+2)}\|x^* - x_0\|^{\alpha_0+2} + \frac{K_0(\alpha_0+2)}{2(\alpha_0+1)}\|x^* - x_0\|^{\alpha_0+2} \\
\leq \ & \frac{K_0(2\alpha_0^2 + 9\alpha_0 + 8)}{2(\alpha_0+1)(\alpha_0+2)}\|x^* - x_0\|^{\alpha_0+2}.
\end{aligned}
$$

Thanks to (5), we obtain $Z_0(x^*) \in \mathbb{B}_b(0)$.
From this result and the definition of ϕ_0 and (4), we get

$$
\begin{aligned}
\text{dist } (x^*, \phi_0(x^*)) \quad & \leq e(P^{-1}(0) \cap \mathbb{B}_\delta(x^*), P^{-1}[Z_0(x^*)]) \\
& \leq M\|Z_0(x^*)\| \\
& \leq \frac{MK_0(2\alpha_0^2 + 9\alpha_0 + 8)}{2(\alpha_0+1)(\alpha_0+2)}\|x^* - x_0\|^{\alpha_0+2}.
\end{aligned} \tag{7}
$$

Since $c > \frac{MK_0(2\alpha_0^2+9\alpha_0+8)}{2(\alpha_0+1)(\alpha_0+2)}$, one can find $\lambda \in]0,1[$ such that $c(1-\lambda) \geq \frac{MK_0(2\alpha_0^2+9\alpha_0+8)}{2(\alpha_0+1)(\alpha_0+2)}$.

Hence,

$$\text{dist}\,(x^*, \phi_0(x^*)) \leq c(1-\lambda)\|x^* - x_0\|^{\alpha_0+2}. \tag{8}$$

By setting $r = r_0 = c\|x^* - x_0\|^{\alpha_0+2}$, condition (a) of Lemma 2 is fulfilled. Let us observe that from (5), $r_0 \leq \delta \leq a$. For $x \in \mathbb{B}_\delta(x^*)$, using (5), we have

$$\begin{aligned}
\|Z_0(x)\| &= \|f(x^*) + \frac{1}{2}\Big(\nabla f(x^*) + \nabla f(x)\Big)(x - x^*) \\
&\quad - f(x_0) - \frac{1}{2}\Big(\nabla f(x_0) + \nabla f(x)\Big)(x - x_0)\| \\
&\leq \|f(x^*) - f(x) - \nabla f(x)(x^* - x) - \frac{1}{2}\nabla^2 f(x)(x - x^*)^2\| \\
&\quad + \|f(x) - f(x_0) - \nabla f(x_0)(x - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x - x_0)^2\| \\
&\quad + \frac{1}{2}\|\nabla f(x^*) - \nabla f(x) - \nabla^2 f(x)(x^* - x)\|.\|x^* - x\| \\
&\quad + \frac{1}{2}\|\nabla f(x) - \nabla f(x_0) - \nabla^2 f(x_0)(x - x_0)\|.\|x - x_0\| \\
&\leq K_0\|x^* - x\|^2 \int_0^1 (1-t)\Big[\|x + t(x^* - x) - x^*\|^{\alpha_0} + \|x^* - x\|^{\alpha_0}\Big]dt \\
&\quad + K_0\|x - x_0\|^2 \int_0^1 (1-t)\Big[\|x_0 + t(x - x_0) - x^*\|^{\alpha_0} + \|x^* - x_0\|^{\alpha_0}\Big]dt \\
&\quad + \frac{1}{2}K_0\|x^* - x\|^2 \int_0^1 \Big[\|(1-t)(x - x^*)\|^{\alpha_0} + \|x^* - x\|^{\alpha_0}\Big]dt \\
&\quad + \frac{1}{2}K_0\|x - x_0\|^2 \int_0^1 \Big[\|tx + (1-t)x_0 - x^*\|^{\alpha_0} + \|x^* - x_0\|^{\alpha_0}\Big]dt \\
&\leq K_0\|x^* - x_0\|^{\alpha_0+2} \int_0^1 \Big[(1-t)^{\alpha_0+1} + 1 - t\Big]dt + K_0\|x - x_0\|^2\delta^{\alpha_0} \\
&\quad \int_0^1 2(1-t)dt + \frac{K_0}{2}\|x^* - x\|^{\alpha_0+2} \int_0^1 \Big[(1-t)^{\alpha_0} + 1\Big]dt \\
&\quad + \frac{K_0}{2}\|x - x_0\|^2\delta^{\alpha_0} \int_0^1 2dt \\
&\leq \frac{K_0(\alpha_0+4)}{2(\alpha_0+2)}\|x^* - x\|^{\alpha_0+2} + K_0\delta^{\alpha_0}\|x - x_0\|^2 \\
&\quad + \frac{K_0(\alpha_0+2)}{2(\alpha_0+1)}\|x^* - x\|^{\alpha_0+2} + K_0\delta^{\alpha_0}\|x - x_0\|^2 \\
&\leq \frac{18\alpha_0^2 + 57\alpha_0 + 40}{2(\alpha_0+2)(\alpha_0+1)}K_0\delta^{\alpha_0+2} < b.
\end{aligned}$$

It follows that for all $x', x'' \in \mathbb{B}_{r_0}(x^*)$, we have

$$
\begin{aligned}
& e(\phi_0(x') \cap \mathbb{B}_{r_0}(x^*), \phi_0(x'')) \\
\leq\ & e(\phi_0(x') \cap \mathbb{B}_{\delta}(x^*), \phi_0(x'')) \\
\leq\ & M\|Z_0(x') - Z_0(x'')\| \\
\leq\ & \frac{M}{2}\Big[\|\nabla f(x^*) - \nabla f(x_0)\|.\|x' - x''\| + \|\nabla f(x') - \nabla f(x'')\|.\|x_0 - x^*\|\Big] \\
\leq\ & \frac{M}{2}\Big[\|\nabla f(x^*) - \nabla f(x_0) - \nabla^2 f(x_0)(x^* - x_0)\|\,\|x' - x''\| \\
& +\|\nabla f(x') - \nabla f(x'') - \nabla^2 f(x'')(x' - x'')\|\,\|x_0 - x^*\| \\
& +\|\nabla^2 f(x'') - \nabla^2 f(x_0)\|\,\|x' - x''\|\,\|x_0 - x^*\|\Big] \\
\leq\ & \frac{M}{2}\Big[\|x^* - x_0\| \int_0^1 \|\nabla^2 f(tx^* + (1-t)x_0) - \nabla^2 f(x^*) + \nabla^2 f(x^*) \\
& -\nabla^2 f(x_0)\|dt\|x' - x''\| + \|x' - x''\| \int_0^1 \|\nabla^2 f(tx' + (1-t)x'') \\
& -\nabla^2 f(x^*) + \nabla^2 f(x^*) - \nabla^2 f(x'')\|dt.\|x^* - x_0\| \\
& +\|\nabla^2 f(x'') - \nabla^2 f(x^*) + \nabla^2 f(x^*) - \nabla^2 f(x_0)\|.\|x' - x''\|.\|x_0 - x^*\|\Big] \\
\leq\ & \frac{M}{2}\Big[K_0\|x^* - x_0\| \int_0^1 (\|tx^* + (1-t)x_0 - x^*\|^{\alpha_0} + \|x^* - x_0\|^{\alpha_0})dt \\
& \|x' - x''\| + K_0\|x' - x''\| \int_0^1 (\|tx' + (1-t)x'' - x^*\|^{\alpha_0} \\
& +\|x^* - x''\|^{\alpha_0})dt\|x^* - x_0\| + \|x' - x''\|\,\|x_0 - x^*\| \\
& (\|\nabla^2 f(x'') - \nabla^2 f(x^*)\| + \|\nabla^2 f(x^*) - \nabla^2 f(x_0)\|)\Big] \\
\leq\ & \frac{M}{2}\Big[K_0\|x^* - x_0\|^{\alpha_0+1} \int_0^1 [(1-t)^{\alpha_0} + 1]dt.\|x' - x''\| \\
& +K_0\|x' - x''\| \int_0^1 2\delta^{\alpha_0} dt\|x^* - x_0\| \\
& +\|x' - x''\|.\|x_0 - x^*\|(K_0\|x'' - x^*\|^{\alpha_0} + K_0\|x^* - x_0\|^{\alpha_0})\Big] \\
\leq\ & \frac{M}{2}\|x' - x''\|\Big[\frac{K_0(\alpha_0 + 2)}{\alpha_0 + 1}\|x^* - x_0\|^{\alpha_0+1} + 2K_0\delta^{\alpha_0}\|x_0 - x^*\| \\
& +K_0\|x'' - x^*\|^{\alpha_0}\|x_0 - x^*\| + K_0\|x^* - x_0\|^{\alpha_0+1}\Big] \\
\leq\ & \frac{MK_0(5\alpha_0 + 6)\delta^{\alpha_0+1}}{2(\alpha_0 + 1)}\|x' - x''\|.
\end{aligned}
$$

Without loss of generality, we can choose δ such that $\delta < \left(\frac{2\lambda(\alpha_0+1)}{MK_0(5\alpha_0+6)}\right)^{\frac{1}{\alpha_0+1}}$, thus condition (b) of Lemma 2 is satisfied. Since both conditions of Lemma 2 are fulfilled, we can deduce that ϕ_0 has a fixed point $x_1 \in \mathbb{B}_{r_0}(x^*)$, that is

$$\|x_1 - x^*\| \leq c\|x_0 - x^*\|^{\alpha_0+2}.$$

Proceeding by induction, keeping $\eta_0 = x^*$ and setting $r_k = c\|x_k - x^*\|^{\alpha_0+2}$, we have the existence of a fixed point x_{k+1} for ϕ_k, which is an element of $\mathbb{B}_{r_k}(x^*)$. Then

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^{\alpha_0+2}. \tag{9}$$

In others words, (x_k) is superquadratically convergent to x^* then the proof of Theorem 1 is complete.

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