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Value Sharing Of Entire Functions*

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Abstract

In this paper, we study the uniqueness of entire functions and prove the following theorem. Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, k two positive integers with $n \geq 5k+8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM. Then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$.

1 Introduction and Results

By a meromorphic function we shall always mean a function that is meromorphic in the open complex plane C. It is assumed that the reader is familiar with the notations of value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in [6]. For a constant a, we define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 1/(f-a))}{T(r,f)}.$$

Let a be a finite complex number, and k a positive integer. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Let a be a complex number,we say f and g share the value a CM, if $f-a$ and $g-a$ assume the same zeros with the same multiplicity. We say f and g share the value a IM, if $f-a$ and $g-a$ assume the same zeros ignoring multiplicity.

Hayman and Clunie proved the following result.

THEOREM A ([7, 3]). Let f be a transcendental entire function, $n \geq 1$ a positive integer. Then $f^nf' = 1$ has infinitely many solutions.

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In 1997, Yang and Hua obtained a unicity theorem corresponding to the above result.

THEOREM B ([11]). Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

In 2000, Xu and Qiu replaced the CM shared value by an IM shared value in Theorem B and proved the following result.

THEOREM C ([10]). Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 12$, and let $a \neq 0$ be a finite constant. If $f^n f'$ and $g^n g'$ share a IM, then either $f(z) = c_1 e^{-cz}$, $g(z) = c_2 e^{cz}$, where c_1, c_2, c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

REMARK: In fact, in [10], Xu and Qiu only considered the situation that f and g were transcendental entire functions, and ignored the situation that f and g were polynomials. For more related results, the reader can refer to [8] or [1].

Chen [2] and Wang [9] extended Theorem A by proving the following theorem.

THEOREM D ([2, 9]). Let f be a transcendental function, n, k two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.

Naturally we ask by Theorem A and Theorem B whether there exists a corresponding unicity theorem to Theorem D ? In 2002, Fang gave a positive answer to the above question and proved the following result.

THEOREM E ([4]). Let $f(z)$ and $g(z)$ be two nonconstant entire functions, n, k two positive integers with $n > 2k+4$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM. Then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$.

It is natural to ask the following question: is it possible to relax the nature of sharing value from CM to IM in Theorem E ? In this paper, we answer the question by proving the following theorem.

THEOREM 1. Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, k two positive integers with $n \geq 5k+8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM. Then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$.

REMARK: When $k = 1$ in Theorem 1, it is Theorem B. So Theorem 1 is also an improvement of Theorem B.

2 Some Lemmas

The following Lemmas are needed in the proof of Theorem 1.

LEMMA 1 ([6, 12]). Let $f(z)$ be a transcendental entire function, k a positive integer, and let c be a nonzero finite complex number. Then

$$\begin{aligned} T(r,f) &\le N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\ &\le N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\,, \end{aligned}$$

where $N_0(r,\frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

LEMMA 2 ([6, 12]). Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f), i=1,2$. Then

$$T(r,f)\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f)\,.$$

LEMMA 3 ([13]). Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Then

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{f-1}\right) &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\,,\\ \overline{N}_L\left(r,\frac{1}{g-1}\right) &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,g)\,, \end{aligned}$$

where $\overline{N}_L(r,\frac{1}{f-1})$ denotes the counting function for 1-points of both f and g about which f has larger multiplicity than g, with multiplicity being not counted.

LEMMA 4 ([14]). Let f be a nonconstant meromorphic function, k be a positive integer, then

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\,,$$

where $N_p(r,\frac{1}{f^{(k)}})$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m\le p$ and p times if $m>p$. Clearly $\overline{N}(r,\frac{1}{f^{(k)}})=N_1(r,\frac{1}{f^{(k)}})$.

LEMA 5. Let $F(z)$ and $G(z)$ be two transcendental entire functions such that $\Theta(0,F)>\frac{5k+6}{5k+7}$, $\Theta(0,G)>\frac{5k+6}{5k+7}$. If $F(z)^{(k)}$ and $G(z)^{(k)}$ share the value 1 IM, then either $F(z)^{(k)}G(z)^{(k)}\equiv 1$ or $F\equiv G$.

PROOF. Set

$$\Phi=\frac{F^{(k+2)}}{F^{(k+1)}}-2\frac{F^{(k+1)}}{F^{(k)}-1}-\frac{G^{(k+2)}}{G^{(k+1)}}+2\frac{G^{(k+1)}}{G^{(k)}-1}\,.$$

Suppose that $\Phi\not\equiv 0$. If z_0 is a common simple zero of $F(z)^{(k)}-1$ and $G^{(k)}-1$, by a simple computation, we know that z_0 is a zero of Φ. Thus we have

$$N_E^{1)}\left(r,\frac{1}{F^{(k)}-1}\right)\le \overline{N}\left(r,\frac{1}{\Phi}\right)\le T(r,\Phi)+O(1)\le N(r,\Phi)+S(r,F)+S(r,G)\,. \quad (1)$$

By our assumption, we obtain

$$\begin{aligned} N(r,\Phi) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{G^{(k)}-1}\right) \\ &\quad +N_0\left(r,\frac{1}{F^{(k+1)}}\right)+N_0\left(r,\frac{1}{G^{(k+1)}}\right). \end{aligned} \tag{2}$$

Note that

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{G^{(k)}-1}\right) \\ \leq\ & N_E^{1)}\left(r,\frac{1}{F^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{F^{(k)}-1}\right)+N\left(r,\frac{1}{G^{(k)}-1}\right) \\ \leq\ & N_E^{1)}\left(r,\frac{1}{F^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{F^{(k)}-1}\right)+T(r,G)+O(1). \end{aligned} \tag{3}$$

By Lemma 1, we have

$$T(r,F) \leq (k+1)\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)-N_0\left(r,\frac{1}{F^{(k+1)}}\right)+S(r,F) \tag{4}$$

$$T(r,G) \leq (k+1)\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G^{(k)}-1}\right)-N_0\left(r,\frac{1}{G^{(k+1)}}\right)+S(r,G). \tag{5}$$

Thus we deduce from (3), (4) and (5) that

$$\begin{aligned} T(r,F)+T(r,G) &\leq (k+1)\overline{N}\left(r,\frac{1}{F}\right)+(k+1)\overline{N}\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F^{(k)}-1}\right) \\ &\quad +\overline{N}_L\left(r,\frac{1}{F^{(k)}-1}\right)+T(r,G)-N_0\left(r,\frac{1}{F^{(k+1)}}\right) \\ &\quad -N_0\left(r,\frac{1}{G^{(k+1)}}\right)+S(r,F)+S(r,G). \end{aligned} \tag{6}$$

So by Lemma 3 and (1), (2) and (6), we have

$$\begin{aligned} T(r,F) &\leq (k+2)\overline{N}\left(r,\frac{1}{F}\right)+(k+2)\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F^{(k)}-1}\right) \\ &\quad +\overline{N}_L\left(r,\frac{1}{G^{(k)}-1}\right)+S(r,F)+S(r,G) \\ &\leq (k+2)\overline{N}\left(r,\frac{1}{F}\right)+(k+2)\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F^{(k)}}\right) \\ &\quad +\overline{N}\left(r,\frac{1}{G^{(k)}}\right)+S(r,F)+S(r,G). \end{aligned} \tag{7}$$

By Lemma 4, we have

$$\overline{N}\left(r,\frac{1}{F^{(k)}}\right) \leq N_{k+1}\left(r,\frac{1}{F}\right)+S(r,F) \leq (k+1)\overline{N}\left(r,\frac{1}{F}\right)+S(r,F). \tag{8}$$

So by (7) and (8), we have

$$T(r,F) \le (3k+4)\overline{N}\left(r,\frac{1}{F}\right) + (2k+3)\overline{N}\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G)\,. \tag{9}$$

Similarly we have

$$T(r,G) \le (3k+4)\overline{N}\left(r,\frac{1}{G}\right) + (2k+3)\overline{N}\left(r,\frac{1}{F}\right) + S(r,F) + S(r,G)\,. \tag{10}$$

So by (9) and (10), we have

$$T(r,F) + T(r,G) \le (5k+7)\overline{N}\left(r,\frac{1}{F}\right) + (5k+7)\overline{N}\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G)\,.$$

So

$$[(5k+7)\Theta(0,F) - 5k - 6]T(r,F) + [(5k+7)\Theta(0,G) - 5k - 6]T(r,G) \le S(r,F) + S(r,G)\,.$$

Thus we obtain a contradiction from the condition. Hence we have $\Phi \equiv 0$, that is

$$\frac{F^{(k+2)}}{F^{(k+1)}} - 2\frac{F^{(k+1)}}{F^{(k)}-1} \equiv \frac{G^{(k+2)}}{G^{(k+1)}} - 2\frac{G^{(k+1)}}{G^{(k)}-1}\,.$$

By solving this we obtain

$$\frac{1}{F^{(k)}-1} = \frac{bG^{(k)}+a-b}{G^{(k)}-1}\,,$$

where a, b are two constants. Next we consider three cases.

Case 1 $b \ne 0$, $a = b$. So we obtain that $G^{(k)} \ne 0$. Thus there exists an entire function h such that $G^{(k)} = e^h$ and

$$F^{(k)} = 1 + \frac{1}{b} - \frac{1}{b}e^{-h}\,.$$

If $b = -1$, then $F^{(k)}G^{(k)} \equiv 1$. If $b \ne -1$, then $F^{(k)} - (1+\frac{1}{b}) = -\frac{1}{b}e^{-h} \ne 0$. And thus we deduce from Lemma 1 that

$$T(r,F) \le (k+1)\overline{N}(r,\frac{1}{F}) + S(r,F) \le (k+1)(1-\Theta(0,F))T(r,F) + S(r,F)\,,$$

that is

$$[(k+1)\Theta(0,F) - k]T(r,F) \le S(r,F)\,.$$

Hence we deduce a contradiction from the assumption.

Case 2. $b \ne 0$, $a \ne b$. Then we have $G^{(k)} + \frac{a-b}{b} \ne 0$. From Lemma 1 we deduce

$$T(r,G) \le (k+1)\overline{N}\left(r,\frac{1}{G}\right) + S(r,G)\,.$$

By using the argument as in Case 1, we get a contradiction.

Case 3. $b=0$, $a\neq 0$. Then we obtain

$$F=\frac{1}{a}G+P(z)\,,$$

where $P(z)$ is a polynomial. If $P(z)\not\equiv 0$, then by Lemma 2 we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-P}\right)+S(r,F)=\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F) \\ &\leq [1-\Theta(0,F)]T(r,F)+[1-\Theta(0,G)]T(r,G)+S(r,F)\,. \end{aligned}$$

Obviously we have $T(r,F)=T(r,G)+S(r,F)$. Hence we get

$$[\Theta(0,F)+\Theta(0,G)-1]T(r,F)\leq S(r,F)\,.$$

Thus we deduce that $T(r,F)\leq S(r,F)$, a contradiction. Therefore we deduce that $P(z)\equiv 0$, that is

$$F=\frac{1}{a}G\,.$$

If $a\neq 1$, then by $F^{(k)}$, $G^{(k)}$ share the value 1 IM, we deduce that $G^{(k)}\not\equiv 1$. Next we can deduce a contradiction as in Case 2. Thus we get $a=1$, that is $F\equiv G$. The proof of the Lemma is complete.

LEMMA 6 ([5]). Let $f(z)$ be a nonconstant entire function, and let $k\geq 2$ be a positive integer. If $ff^{(k)}\neq 0$, then $f=e^{az+b}$, where $a\neq 0, b$ are constants.

3 Proof of Theorem 1

We only prove the case of $k\geq 2$ from Theorem B. Let $F=f^n$, $G=g^n$. Then by the assumptions we obtain

$$\Theta(0,F) \geq \frac{n-1}{n}>\frac{5k+6}{5k+7}\,, \tag{11}$$

$$\Theta(0,G) \geq \frac{n-1}{n}>\frac{5k+6}{5k+7}\,. \tag{12}$$

Considering $F^{(k)}=[f^n]^{(k)}$, $G^{(k)}=[g^n]^{(k)}$, we obtain that $F^{(k)}$, $G^{(k)}$ share the value 1 IM. Hence by (11), (12) and Lemma 5 we deduce that $F(z)^{(k)}G(z)^{(k)}\equiv 1$ or $F\equiv G$. Next we consider two cases.

Case 1. $F(z)^{(k)}G(z)^{(k)}\equiv 1$, that is

$$[f^n]^{(k)}[g^n]^{(k)}\equiv 1\,.$$

Obviously, $f\neq 0$ and $g\neq 0$. In fact, suppose that f has a zero z_0. Then z_0 is a zero of $[f^n]^{(k)}$. Thus z_0 is a pole of $[g^n]^{(k)}$, which contradicts that g is an entire function. Hence $f\neq 0$, $g\neq 0$. On the other hand, we get by f, g are entire functions that

$$[f^n]^{(k)}\neq 0, [g^n]^{(k)}\neq 0\,.$$

Then by Lemma 6, we get that $f = c_1e^{cz}$, $g = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.

Case 2. $F \equiv G$, that is $f^n \equiv g^n$. Hence we get $f = tg$, where t is a constant satisfying $t^n = 1$. Thus Theorem 1 is proved.

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