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Spline Solution Of Some Linear Boundary Value Problems*

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Abstract

In this paper, a spline collocation method using spline interpolants is developed and analyzed for approximating solutions of some general linear boundary value problems. It is observed that the method developed in this paper when applied to some examples is better than other collocation and spline methods given in the literature.

1 Introduction

Boundary value problems (BVPs) can be used to model several physical phenomena. For example, when an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability sets in as ordinary convection, the ordinary differential equation is sixth order. When the instability sets in as overstability, it is modeled by an eighth-order ordinary differential equation [6]. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modeled by tenth-order boundary value problem. When instability sets in as overstability, it is modeled by twelfth-order boundary value problem [6].

In this paper, we study a spline collocation method allowing to compute the numerical solution of BVPs described by a differential equation and boundary conditions of the form

$$y^{(r)}(x) + p(x)y(x) = g(x), \quad \forall x \in [a, b], \tag{1}$$

$$y^{(m)}(a) = \alpha_m,\ m = 0, \ldots, r_2 - 1,\ y^{(m)}(b) = \beta_m,\ m = 0, \ldots, r_1 - 1, \tag{2}$$

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where $p(x)$ and $g(x)$ are given continuous functions defined in the bounded interval $[a, b]$, α_i $(i = 0, \ldots, r_2 - 1)$, and β_i $(i = 0, \ldots, r_1 - 1)$ are real finite constants, with

$$r_2 = \lfloor \frac{r}{2} \rfloor = \max\left\{n \in \mathbb{N},\ n \leq \frac{r}{2}\right\} \quad \text{and} \quad r_1 = \lceil \frac{r}{2} \rceil = \min\{n \in \mathbb{N},\ n \geq \frac{r}{2}\}.$$

Agarwal's book [1] contains theorems which detail the conditions for the existence and the uniqueness of solutions of such BVPs.

The spline collocation methods have been extensively applied in numerical ordinary differential equations due to their easy implementation and high-order accuracy. However, in a series of papers by Siddiqi and Twizell [8, 9, 10] linear BVPs of orders 12, 10, and 8 were solved using thirteen, eleventh and nonic degree splines respectively, where some unexpected results were obtained near the boundaries of the interval. Therefore, to remedy these drawbacks, we propose a new collocation method using a spline interpolant which satisies the same boundary conditions. It is to be noted that numerical examples, given in Section 3, indicate that no such unexpected situation is observed near the boundaries of the interval when we use our method.

The paper is organized as follows. Section 2 is devoted to the spline collocation method, based on a spline interpolant, for BVPs using a spline interpolant. Next, the error bound of the spline solution is analyzed. Finally, in order to compare this method with the other ones developed in the literature, we give in Section 3 three numerical examples.

2 Collocation Method using a Spline Interpolant

2.1 Spline Interpolant

Collocation method is often presented as a generalization of interpolation. More specifically, if the differential operator is reduced to the identity operator, the collocation method is reduced to interpolation. Moreover, the order of convergence of the collocation method is often related to that of the interpolant in the same approximation space.

In this section, we define a spline interpolant S of degree $r + 1$ satisfying boundary conditions (2) with optimal approximation order. To do this we consider the uniform grid partition

$$\Delta = \{a = x_{-r-1} = \ldots = x_0 < x_1 < \ldots < x_{n-1} < x_n = \ldots = x_{n+r+1} = b\},$$

of the interval $I = [a, b]$, where $x_i = a + ih,\ 0 \leq i \leq n$, and $h = (b - a)/n$. Let $B_i,\ i = -r - 1, \ldots, n - 1$, be the B-splines of degree $r + 1$ associated with Δ. It is well known that these B-splines form a basis of the space $S^r_{r+1}(I, \Delta) = \{s \in \mathcal{C}^r : s|_{[x_i, x_{i+1}]}$ is a polynomial of degree $\leq r + 1\}$.

THEOREM 1. Let y be the exact solution of the problem (1) with boundary conditions (2), then there exists a unique spline interpolant $S \in S^r_{r+1}(I, \Delta)$ of y satisfying

$$\begin{aligned} S^{(m)}(t_0) &= y^{(m)}(t_0) = \alpha_m,\ m = 0, \ldots, r_2 - 1, & (3)\\ S(t_i) &= y(t_i),\quad i = 1, \ldots, n + 1 & (4)\\ S^{(m)}(t_{n+2}) &= y^{(m)}(t_{n+2}) = \beta_m,\ m = 0, \ldots, r_1 - 1, & (5) \end{aligned}$$

where $t_0 = x_0,\ t_i = (x_i+x_{i-1})/2,\ i = 1, \ldots, n,\ t_{n+1} = x_{n-1},\ t_{n+2} = x_n$ if r is odd, and $t_0 = x_0,\ t_1 = (x_0 + x_1)/2,\ t_{i+1} = x_i,\ i = 1, \ldots, n-1,\ t_{n+1} = (x_{n-1} + x_n)/2,\ t_{n+2} = x_n$ if r is even.

PROOF. Let $S = \sum_{j=-r-1}^{n-1} c_j B_j$ be a spline in $\mathcal{S}_{r+1}^r([a,b],\tau)$ that satisfies the conditions (3) and (5). Since

$$\frac{(\cdot - \omega)^{r+2-\nu}}{(r+2-\nu)!} = \sum_{j=-r-1}^{n-1} \frac{(-D)^{\nu-1}\psi_j(\omega)}{(r+1)!} B_j, \quad \nu = 1, \ldots, r+2,$$

where $\psi_j(\omega) = (x_{j+1} - \omega) \cdots (x_{j+r+1} - \omega)$, and D is the derivative operator, we have

$$c_j = \sum_{\nu=1}^{j+r+2} \frac{1}{(r+1)!}(-D)^{r+2-\nu}\psi_j(a)\ y^{(\nu-1)}(a), \quad j = -r-1, \ldots, -r-2+r_2,$$

$$c_j = \sum_{\nu=1}^{n-j} \frac{1}{(r+1)!}(-D)^{r+2-\nu}\psi_j(b)\ y^{(\nu-1)}(b), \quad j = n-r_1, \ldots, n-1.$$

On the other hand, if we put

$$S(x) = \mu(x) + S_1(x), \tag{6}$$

where $\mu(x) = \sum_{j=-r-1}^{-r-2+r_2} c_j B_j(x) + \sum_{j=n-r_1}^{n-1} c_j B_j(x)$, then the spline S satisfies (4) if and only if

$$S_1(t_i) = \sum_{-r+r_2-1}^{n-r_1-1} c_j\ B_j(t_i) = y(t_i) - \mu(t_i),\ i = 1, \ldots, n+1. \tag{7}$$

As $B_j(t_{j+r-r_2}) > 0,\ -r + r_2 - 1 \le j \le n - r_1 - 1$, we deduce, from Schoenberg-Whitney theorem (see [5]), that there exists a unique vector $(c_j)_{-r+r_2-1}^{n-r_1-1}$ satisfying (7). This completes the proof.

It is well known, see [5], that the interpolation with splines of degree d gives $O(h^{d+1})$ uniform norm errors for the interpolant and $O(h^{d+1-s})$ errors for the s-th derivative of the interpolant. Thus, we have for any $y \in \mathcal{C}^{r+2}([a,b])$

$$\|D^s(y - S)\|_\infty = O(h^{r+2-s}),\ \text{for}\ \ s = 0, \ldots, r. \tag{8}$$

2.2 Spline Collocation Method

We suppose that the exact solution of the BVPs (1) and (2) is of class $\mathcal{C}^{r+2}([a,b])$. Since the interpolatory spline S satisfies (8), it follows from (1) that

$$S^{(r)}(\tau_i) + p(t_i)S(\tau_i) = g(\tau_i) + O(h^2), \quad i = 1, \ldots, n+1, \tag{9}$$

where $\tau_{i+1} = (x_i + x_{i+1})/2,\ i = 0, \ldots, n-1,\ \tau_{n+1} = x_{n-1}$. Then, the spline collocation method presented in this section consists in finding a spline

$$\widetilde{S}(x) = \mu(x) + \sum_{j=-r+r_2-1}^{n-1-r_1} \widetilde{c}_j B_j(x), \tag{10}$$

which satisfies

$$\widetilde{S}^{(r)}(\tau_i)+p(\tau_i)\widetilde{S}(\tau_i)=g(\tau_i),\quad i=1,\dots,n+1. \tag{11}$$

Taking $C=[c_{-r+r_2-1},\dots,c_{n-r_1-1}]^T$ and $\widetilde{C}=[\widetilde{c}_{-r+r_2-1},\dots,\widetilde{c}_{n-r_1-1}]^T$, and using equations (9) and (11), we get

$$\begin{aligned}(A_h+B_hP)C &= G+e, && (12)\\ (A_h+B_hP)\widetilde{C} &= G, && (13)\end{aligned}$$

where $G=[g_1\dots,g_{n+1}]$, $e=[e_1\dots,e_{n+1}]$ with $g_i=g(\tau_i)-\mu^{(r)}(\tau_i)-p(\tau_i)\mu(\tau_i),\ e_i=O(h^2),\quad i=1,\dots,n+1$, and P, A_h and B_h are the following $(n+1)\times(n+1)$ matrices:

$$A_h=\left(a_{i,j}^{(h)}\right)_{1\le i,j\le n+1},$$

$$B_h=\left(b_{i,j}^{(h)}\right)_{1\le i,j\le n+1}\ \text{and}\ P=(diag(p(\tau_i))_{1\le i\le n+1},$$

with $a_{i,j}^{(h)}=B^{(r)}_{-r+r_2-2+j}(\tau_i)$ and $b_{i,j}^{(h)}=B_{-r+r_2-2+j}(\tau_i)$.

Let $M_j,\ j=-r-1,\dots,n-1$, be the B-splines of degree $r+1$ associated with the uniform partition

$$X_n=\{0=x_{-r-1}=\dots=x_0<x_1=1<\dots<x_{n-1}=n-1<x_n=\dots=x_{n+r+1}=n\},$$

then

$$B_j(x)=M_j\left(\frac{x-a}{h}\right),\ \forall x\in[a,b],\ \text{and therefore}\ B_j^{(r)}(\tau_i)=\frac{1}{h^r}M_j^{(r)}\left(\frac{\tau_i-a}{h}\right).$$

Taking

$$A=(a_{i,j})_{1\le i,j\le n+1},\quad\text{where}\quad a_{i,j}=M^{(r)}_{-r+r_2-2+j}\left(\frac{\tau_i-a}{h}\right),\ \text{we get}\ A_h=\frac{1}{h^r}A,$$

we deduce that (12) and (13) can be written in the following form

$$(A+h^rB_hP)C=h^r(G+e), \tag{14}$$

$$(A+h^rB_hP)\widetilde{C}=h^rG. \tag{15}$$

In order to determine the bound of $\|C-\widetilde{C}\|_\infty$, we need the following lemma.

LEMMA 1. The matrix A is invertible.

PROOF. It suffices to prove that for all $D=[d_1,\dots,d_{n+1}]^T\in\mathbb{R}^{n+1}$ such that $AD=0$, we have $D=0$. Indeed, If we put

$$z(x)=\sum_{j=-r+r_2-1}^{n-1-r_1} d_{j+r-r_2+2}\ B_j(x),$$

then, $z^{(r)}(\tau_i) = 0, \quad \forall\ i = 1, \ldots, n+1$. On the other hand, using the fact that $z^{(r)}$ is a continuous spline function of degree 1, we deduce that $z^{(r)}(x) = \alpha x + \beta,\ \forall\ x \in [x_{n-1}, x_n]$. As

$$z^{(r)}(\tau_n) = z^{(r)}\left(\frac{x_{n-1}+x_n}{2}\right) = 0 \quad \text{and} \quad z^{(r)}(\tau_{n+1}) = z^{(r)}(x_{n-1}) = 0,$$

we have $z^{(r)}(x) = 0,\ \forall\ x \in [x_{n-1}, x_n]$. In a similar manner, we can easily prove that $z^{(r)}(x) = 0$ in all the other subintervals of $[a, b]$. Then, we conclude that

$$\begin{cases} z^{(r)}(x) = 0, \text{ if } \ x \in \ [a, b] \\ z^{(m)}(a) = 0,\ m = 0, \ldots, r_1 - 1,\ z^{(m)}(b) = 0,\ m = 0, \ldots, r_2 - 1, \end{cases}$$

which has 0 as unique solution. Consequently $z(x) = 0$ for all $x \in \ [a, b]$, which in turn gives $D = 0$.

PROPOSITION 1. If $h^r \|A^{-1}\|_\infty \|P\|_\infty < 1$, then there exists a unique spline $\widetilde{S}$ that approximates the exact solution y of problem (1) with boundary conditions (2).

PROOF. Assume that $h^r \|A^{-1}\|_\infty \|P\|_\infty < 1$. As $\|B_h\|_\infty \leq 1$, we have

$$h^r \|A^{-1}\|_\infty \|B_h\|_\infty \|P\|_\infty < 1.$$

Consequently, $(I + h^r A^{-1} B_h P)^{-1}$ exists and

$$\|(I + h^r A^{-1} B_h P)^{-1}\|_\infty < \frac{1}{1 - h^r \|A^{-1}\|_\infty \|P\|_\infty}.$$

From (15), we get

$$\widetilde{C} = h^r (I + h^r A^{-1} B_h P)^{-1} A^{-1} G.$$

PROPOSITION 2. If $h^r \|A^{-1}\|_\infty \|P\|_\infty \leq \frac{1}{2}$, then there exists a constant K which depends only on the functions p and g such that

$$\|C - \widetilde{C}\|_\infty \leq K h^2. \tag{16}$$

PROOF. Assume that $h^r \|A^{-1}\|_\infty \|P\|_\infty \leq \frac{1}{2}$, from (14) and (15), we have $C - \widetilde{C} = h^r (I + h^r A^{-1} BP)^{-1} A^{-1} e$. Thus, $\|C - \widetilde{C}\|_\infty < \frac{h^r \|A^{-1}\|_\infty}{1 - h^r \|A^{-1}\|_\infty \|P\|_\infty} \|e\|_\infty$. Since $e = O(h^2)$, there exists a constant K_1 such that $\|e\|_\infty \leq K_1 h^2$. On the other hand,

$$\frac{h^r \|A^{-1}\|_\infty}{1 - h^r \|A^{-1}\|_\infty \|P\|_\infty} \leq \frac{1}{\|P\|_\infty}, \quad \text{we deduce that} \quad \|C - \widetilde{C}\|_\infty \leq \frac{K_1}{\|P\|_\infty}\, h^2.$$

Now, we are in position to prove the main theorem of this section.

THEOREM 2. The spline approximation $\widetilde{S}$ converges quadratically to the exact solution y of the BVP defined by (1) and (2), i.e., $\|y - \widetilde{S}\|_\infty = O(h^2)$.

PROOF. According to (8), there exists a constant K_3 such that $\|y-S\|_\infty \leq K_3 h^{r+2}$. On the other hand we have $S(x)-\widetilde{S}(x) = \sum_{j=-r+r_2-1}^{n-1-r_1} (c_j-\widetilde{c}_j)B_j(x)$. Therefore, by using (16), we get

$$|S(x) - \widetilde{S}(x)| \leq \|C - \widetilde{C}\|_\infty \sum_{j=-r+r_2-1}^{n-1-r_1} B_j(x) \leq \|C - \widetilde{C}\|_\infty \leq Kh^2.$$

As $\|y - \widetilde{S}\|_\infty \leq \|y - S\|_\infty + \|S - \widetilde{S}\|_\infty$, we deduce the stated result.

3 Numerical results

The nonic, eleventh and thirteen spline methods for the solutions of linear BVPs of orders 8, 10 and 12, respectively along with the corresponding errors in absolute value, are illustrated in the following numerical examples.

3.1 Eighth order boundary value problems

For $r = 8$, we have

$$\mu(x) = \sum_{j=-9}^{-6} c_j B_j(x) + \sum_{j=n-4}^{n-1} c_j B_j(x).$$

where the coefficients $c_j,\ -9 \leq j \leq -6$ and $c_j,\ n-4 \leq j \leq n-1$, are computed explicitly.

EXAMPLE 1. Consider the boundary-value problem:

$$\begin{cases} y^{(8)}(x) - y(x) = -8(2x\cos(x) + 7\sin(x)), \quad x \in [-1, 1], \\ y(-1) = 0,\ y^{'}(-1) = 2\sin(1),\ y^{''}(-1) = -4\cos(1) - 2\sin(1),\ y^{(3)}(-1) = 6\cos(1) - 6\sin(1) \\ y(1) = 0,\ y^{'}(1) = 2\sin(1),\ y^{''}(1) = 4\cos(1) + 2\sin(1),\ y^{(3)}(1) = 6\cos(1) - 6\sin(1), \end{cases} \tag{17}$$

for which the theoretical solution is $y(x) = (x^2 - 1)\sin(x)$.

The comparison of the errors in absolute values between the method developed in this paper with those developed by Siddiqi and Twizell [10] and Akram and Siddiqi [2] for $h = \frac{1}{32}$ is shown in Table 1.

Table 1: Comparison of numerical results for the problem 17.

Siddiqi and Twizell [10] $x \in [x_4, x_{n-4}]$	Siddiqi and Twizell Otherwise	Akram and Siddiqi [2]	Our method
$1.20e-005$	$1.61e+004$	$1.02e-008$	$5.0103e-009$

3.2 Tenth-order boundary value problems

For $r = 10$, we have $\mu(x) = \sum_{j=-11}^{-7} c_j B_j(x) + \sum_{j=n-5}^{n-1} c_j B_j(x)$, where the coefficients $c_j,\ -11 \le j \le -7$ and $c_j,\ n-5 \le j \le n-1$ are computed explicitly.

EXAMPLE 2. Consider the following boundary value problem:

$$\begin{cases} y^{(10)}(x) - xy(x) = -(89 + 21x + x^2 - x^3)e^x, \quad x \in [-1, 1], \\ y(-1) = 0,\ y^{'}(-1) = \frac{2}{e},\ y^{''}(-1) = \frac{2}{e},\ y^{(3)}(-1) = 0,\ y^{(4)}(-1) = -\frac{4}{e} \\ y(1) = 0,\ y^{'}(1) = -2e,\ y^{(3)}(1) = -12e,\ y^{(4)}(1) = -20e, \end{cases} \tag{18}$$

for which the theoretical solution is $y(x) = (1 - x^2)e^x$. The errors in absolute values compared with those considered by Siddiqi and Twizell [9] and Akram and Siddiqi [3], corresponding to the developed method for $h = \frac{1}{9}$ are shown in Table 2.

Table 2: Comparison of numerical results for the problem 18.

Siddiqi and Twizell [9] $x \in [x_4, x_{n-4}]$	Siddiqi and Twizell Otherwise	Akram and Siddiqi [3]	Our method
$2.07e-003$	$1.06e+013$	$3.28e-006$	$1.8558e-008$

3.3 Twelfth order boundary value problems

For $r = 12$, we have $\mu(x) = \sum_{j=-13}^{-8} c_j B_j(x) + \sum_{j=n-6}^{n-1} c_j B_j(x)$, where the coefficients $c_j,\ -13 \le j \le -8$ and $c_j,\ n-6 \le j \le n-1$, are computed explicitly.

EXAMPLE 3. Consider the following boundary value problem:

$$\begin{cases} y^{(12)}(x) + xy(x) = -(120 + 23x + x^3)e^x, \quad x \in [0, 1], \\ y(0) = 0,\ y^{'}(0) = 1,\ y^{''}(0) = 0,\ y^{(3)}(0) = -3,\ y^{(4)}(0) = -8,\ y^{(5)}(0) = -15, \\ y(1) = 0,\ y^{'}(1) = -e,\ y^{''}(1) = -4e,\ y^{(3)}(1) = -9e,\ y^{(4)}(1) = -16e,\ y^{(5)}(1) = -25e, \end{cases} \tag{19}$$

The exact solution of the above system is $y(x) = x(1 - x)e^x$. A comparison of the maximum errors (in absolute values) for the problem 19 is summarized in Table 3.

Table 3: Comparison of numerical results for the problem 18, with $n = 22$.

Siddiqi and Twizell [8] $x \in [x_6, x_{n-6}]$	Siddiqi and Twizell Otherwise	Akram and Siddiqi [4]	Our method
$5.58e-003$	$2.65e+024$	$7.38e-009$	$3.7961e-014$

4 Conclusion

Spline collocation method based on spline interpolants is developed for the approximate solution of some general linear BVPs. The method is also proved to be second order convergent. It has been observed that the relative errors (in absolute values), are better than those given by other collocation and spline methods. So, its extension to general nonlinear boundary value problems is under progress.

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