Applied Mathematics E-Notes, 8(2008), 109-114 © ISSN 1607-2510
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The Equivalence Between The T-Stabilities Of Picard-Banach And Mann-Ishikawa Iterations*

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Received 5 February 2007

Abstract

We show that T-stability of Picard-Banach and Mann-Ishikawa iterations are equivalent.

1 Introduction

Let X be a normed space and T a selfmap of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of point from X. Suppose $\{x_n\}$ converges to a fixed point x^* of T. Let $\{\xi_n\}$ be an arbitrary sequence in X, and set $\epsilon_n = \|\xi_{n+1} - f(T, \xi_n)\|$ for all $n \in \mathbb{N}$.

DEFINITION 1. [2] If $((\lim_{n\to\infty} \epsilon_n = 0) \Rightarrow (\lim_{n\to\infty} \xi_n = p))$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be *T-stable with respect to T*.

REMARK 1. [2] In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $\xi_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value ξ_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate ξ_2, the value $f(T, \xi_1)$ is computed to yields ξ_2, an approximation of $f(T, \xi_1)$. This computation is continued to obtain $\{\xi_n\}$ an approximate sequence of $\{x_n\}$.

Consider $e_0 = s_0 = t_0 = g_0 = h_0$. The Picard-Banach iteration is given by

$$b_{n+1} = Tb_n. \tag{1}$$

The two most popular iteration procedures for obtaining fixed points of T, when the Banach principle fails, are Mann iteration [3], defined by

$$e_{n+1} = (1 - \alpha_n)e_n + \alpha_n Te_n, \tag{2}$$

and Ishikawa iteration [1], defined by

$$\begin{aligned} s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n Tt_n, \\ t_n &= (1 - \beta_n)s_n + \beta_n Ts_n. \end{aligned} \tag{3}$$

*Mathematics Subject Classifications: 47H10

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We have $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\alpha_n\}$ usually satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{4}$$

Recently, the equivalence between the T-stabilities of Mann and Ishikawa iterations was shown in [6]. In this note we shall prove the equivalence between T-stabilities of (1), (2) and (3).

2 The Equivalence between T-Stabilities

Let X be a normed space and $T : X \to X$ a map. Let $\{u_n\}, \{p_n\}, \{x_n\}, \{y_n\} \subset X$ be such that $u_0 = p_0 = x_0 = y_0$, and consider

$$\begin{aligned}\varepsilon_n &:= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\|,\\ \delta_n &:= \|p_{n+1} - T p_n\|.\end{aligned}$$

For $\{\beta_n\} \subset [0,1)$, we consider $y_n = (1-\beta_n)x_n + \beta_n T x_n$, and

$$\xi_n := \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\|.$$

DEFINITION 2. Definition 1 gives:

(i) The Ishikawa iteration (3), is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, $\forall \{x_n\} \subset X$ given, we have

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*.$$

The Mann iteration is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$, $\forall \{u_n\} \subset X$ given, we have

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| = 0 \Rightarrow \lim_{n\to\infty} u_n = x^*.$$

(ii) The Picard iteration is said to be T-stable if and only if for all $\{p_n\} \subset X$ given, we have

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|p_{n+1} - T p_n\| = 0 \Rightarrow \lim_{n\to\infty} p_n = x^*.$$

It is obvious that for $\alpha_n := 0, \forall n \in \mathbb{N}$, $\beta_n := 0, \forall n \in \mathbb{N}$, one obtains $\xi_n = \varepsilon_n = \delta_n$.

THEOREM 1. Let X be a normed space and $T : X \to X$ a map. If

$$\lim_{n\to\infty} \|p_n - T p_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_n - T u_n\| = 0, \tag{5}$$

then the following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$, the Mann iteration is T-stable,

(ii) the Picard iteration is T-stable.

PROOF. $(i) \Rightarrow (ii)$. Take $\lim_{n\to\infty} \delta_n = 0$. Observe that

$$\begin{aligned}
\varepsilon_n &= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| \\
&\le \|u_{n+1} - Tu_n\| + (1-\alpha_n)\|u_{n+1} - u_n\| + (1-\alpha_n)\|u_{n+1} - Tu_n\| \\
&\le (2-\alpha_n)\|u_{n+1} - Tu_n\| + (1-\alpha_n)\|u_{n+1} - u_n\| \\
&\le (2-\alpha_n)\|u_{n+1} - Tu_n\| + (1-\alpha_n)(\|u_{n+1} - Tu_n\| + \|u_n - Tu_n\|) \\
&= (3-2\alpha_n)\|u_{n+1} - Tu_n\| + (1-\alpha_n)\|u_n - Tu_n\| \\
&= (3-2\alpha_n)\delta_n + (1-\alpha_n)\|u_n - Tu_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. We know from (i) that if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} u_n = x^*$, thus we have shown that if $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|u_{n+1} - Tu_n\| = 0$, then $\lim_{n\to\infty} u_n = x^*$.

For $(ii) \Rightarrow (i)$, take $\lim_{n\to\infty} \varepsilon_n = 0$. Observe that

$$\begin{aligned}
\delta_n &= \|p_{n+1} - Tp_n\| \\
&\le \|p_{n+1} - (1-\alpha_n)p_n - \alpha_n Tp_n\| + (1-\alpha_n)\|p_n - Tp_n\| \\
&\le \varepsilon_n + (1-\alpha_n)\|p_n - Tp_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. We know from (ii) that if $\lim_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} p_n = x^*$, thus we have shown that if $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|p_{n+1} - (1-\alpha_n)p_n - \alpha_n Tp_n\| = 0$, then $\lim_{n\to\infty} p_n = x^*$.

REMARK 2. Note that no boundedness condition is needed in the above result. Note that $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$ is used in order to prove that $\lim_{n\to\infty} \varepsilon_n = 0$, hence can not be avoided. Analogously, $\lim_{n\to\infty} \|p_n - Tp_n\| = 0$ is used in order to prove that $\lim_{n\to\infty} \delta_n = 0$, hence can not be avoided.

THEOREM 2. Let X be a normed space and $T : X \to X$ a map with bounded range. If

$$\lim_{n\to\infty} \|p_n - Tp_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

then the following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$, satisfying (4), the Ishikawa iteration is T-stable,

(ii) the Picard iteration is T-stable.

PROOF. Let

$$M := \max\left\{\sup_{x\in X}\{\|T(x)\|\}, \|x_0\|\right\}.$$

Since T has bounded range, we have $M < \infty$.

We shall prove that $(i) \Rightarrow (ii)$. Take $\lim_{n\to\infty} \delta_n = 0$. Observe that

$$\begin{aligned}
\xi_n &= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| \\
&\le \|x_{n+1} - Tx_n\| + \|(1-\alpha_n)x_n - \alpha_n T y_n + Tx_n\| \\
&= \|x_{n+1} - Tx_n\| + \|(1-\alpha_n)x_n - \alpha_n T y_n + Tx_n - \alpha_n Tx_n + \alpha_n Tx_n\| \\
&\le \|x_{n+1} - Tx_n\| + (1-\alpha_n)\|x_n - Tx_n\| + \alpha_n \|Tx_n - Ty_n\| \\
&= \delta_n + (1-\alpha_n)\|x_n - Tx_n\| + 2\alpha_n M \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Condition (i) assures that $\lim_{n\to\infty} \xi_n = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*$. Thus, for a $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0,$$

we have shown that $\lim_{n\to\infty} x_n = x^*$.

Conversely, we prove $(ii) \Rightarrow (i)$. Take $\lim_{n\to\infty} \xi_n = 0$. Observe that

$$\begin{aligned}
\delta_n &= \|p_{n+1} - Tp_n\| \\
&\le \|p_{n+1} - (1-\alpha_n)p_n - \alpha_n T y_n\| + \|(1-\alpha_n)p_n + \alpha_n T y_n - Tp_n\| \\
&\le \|p_{n+1} - (1-\alpha_n)p_n - \alpha_n T y_n\| + \alpha_n (\|p_n\| + \|Ty_n\|) + \|p_n - Tp_n\| \\
&\le \varepsilon_n + \alpha_n (\|p_n\| + M) + \|p_n - Tp_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Note that $\lim_{n\to\infty} \|p_n - Tp_n\| = 0$ and using the boundedness of $\{Tp_n\}$ we obtain the boundedness of $\{p_n\}$. Condition (ii) assures that

$$\lim_{n\to\infty} \delta_n = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*.$$

Thus, for a $\{p_n\}$ satisfying $\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \|p_{n+1} - (1-\alpha_n)p_n - \alpha_n T y_n\| = 0$, we have shown that $\lim_{n\to\infty} p_n = x^*$.

Theorems 1 and 2 lead to the following result.

COROLLARY 1. Let X be a normed space and $T : X \to X$ a map with bounded range. If

$$\lim_{n\to\infty} \|p_n - Tp_n\| = 0, \ \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_n - Tu_n\| = 0,$$

then the following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$, satisfying (4), the Ishikawa iteration is T-stable,

(ii) for all $\{\alpha_n\} \subset (0,1)$, satisfying (4), the Mann iteration is T-stable,

(iii) the Picard iteration is T-stable.

3 Applications

The following example is from [2] and [4]. For sake of completeness we give here the whole proof.

EXAMPLE 1. Let $T : [0,1] \to [0,1]$, $Tx = x$.

- [2] Picard iteration converges but is not $T-$stable. Then every point in $(0,1]$ is a fixed point of T. Let b_0 be a point in $(0,1]$, then $b_{n+1}=Tb_n=T^n b_0=b_0$. Thus $\lim_{n\to\infty} b_n=b_0$. Take $p_0=0$ and $p_n=\frac{1}{n}$. Thus

$$\delta_n=|p_{n+1}-Tp_n|=\frac{1}{n\,(n+1)}\to 0,$$

but $\lim_{n\to\infty} p_n=0\neq b_0$.

- [4] Mann iteration converges but is not T-stable. Let e_0 be a point in $(0,1]$, then $e_{n+1}=(1-\alpha_n)\,e_n+\alpha_n e_n=e_n=...=e_0$. Take $u_0=e_0$, $u_n=\frac{1}{n+1}$ to obtain

$$\begin{aligned}\varepsilon_n &= |u_{n+1}-(1-\alpha_n)\,u_n-\alpha_n T u_n|=\left|\frac{1}{n+2}-(1-\alpha_n)\,\frac{1}{n+1}-\alpha_n\frac{1}{n+1}\right|\\ &= \left|\frac{1}{n+2}-\frac{1}{n+1}\right|=\frac{1}{(n+1)\,(n+2)}\to 0,\end{aligned}$$

but $\lim_{n\to\infty} u_n=0\neq e_0$.

EXAMPLE 2. Let $T:[0,\infty)\to[0,\infty)$ be given by $Tx=\frac{x}{3}$. Then the Mann iteration converges to the fixed point of $x^*=0$ but is not $T-$stable, and applying Theorem 1, the Picard iteration is not T-stable while it converges.

(i) Mann iteration converges because the sequence $e_n\to 0$ as we can see:

$$\begin{aligned}e_{n+1} &= (1-\alpha_n)\,e_n+\alpha_n\frac{e_n}{3}=\left(1-\frac{2\alpha_n}{3}\right)e_n\\ &= \prod_{k=1}^{n}\left(1-\frac{2\alpha_k}{3}\right)e_0\leq\exp\left(-\frac{2}{3}\sum_{k=1}^{n}\alpha_k\right)\to 0,\end{aligned}$$

the last inequality is true because $1-x\leq\exp(-x)\,,\forall x\geq 0$, and $\sum\alpha_n=+\infty$ supplied by (4).

(ii) Mann iteration is not T-stable. Take $u_n=\frac{n}{n+1}$, note that $u_n\to 1\neq x^*=0$, and $\varepsilon_n=\|u_{n+1}-(1-\alpha_n)\,u_n-\alpha_n T u_n\|\to 0$ because

$$\begin{aligned}\varepsilon_n &= \left|\frac{n+1}{n+2}-(1-\alpha_n)\,\frac{n}{n+1}-\alpha_n\frac{n}{3(n+1)}\right|\\ &= \frac{3+2\alpha_n n^2+4\alpha_n n}{3\,(n+1)\,(n+2)}.\end{aligned}$$

(iii) Picard iteration converges to fixed point $x^*=0$, because $b_{n+1}=Tb_n=T^n b_0=\frac{b_0}{3^n}\to 0$.

REMARK. Take again $T:[0,\infty)\to[0,\infty)$, $Tx=\frac{x}{3}$, and $x_n=\frac{n}{n+1}$ to note that $\lim_{n\to\infty}\xi_n=0$ and $\lim_{n\to\infty}x_n=1\neq x^*=0$, and to conclude that Ishikawa iteration is not $T-$stable. Remark (analogously to Mann iteration, see also [5]) that it converges while T is a contraction.

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