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Properties Of Solutions To A Generalized Liénard Equation With Forcing Term*

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Abstract

This paper discusses under what conditions the solutions to a generalized Liénard equation $x'' + c(t)f(x)x' + a(t)g(x) = e(t)$ are bounded on $[0, \infty)$ with specified conditions on c, f, a, g and e. Specifically, we shall show that all solutions are bounded whether e is bounded or absolutely integrable. In the bounded case, however, we shall require that e' must be of fixed sign along with the condition that $e'(t)a(t) - e(t)a'(t) \neq 0$. Finally, a brief discussion of L^p-solutions is given under somewhat more restrictive conditions.

In this note we will study in detail some new results concerning the global properties of a generalized Liénard equation of the form

$$x'' + c(t)f(x)x' + a(t)g(x) = e(t). \tag{1}$$

This equation has been well-studied and the results here extend the results of Fonda and Zanolin, Kroopnick, and Nkashama (see [2-6] as well as their excellent lists of references) in which the authors assumed a periodic forcing term in [1] and in [2] the forcing term was 0 or absolutely integrable and the conditions on c and a were somewhat less general. For our purposes, the forcing term will either be absolutely integrable or bounded. It is the bounded case which is new. For our first result we will assume that:

(a) c is a continuous, non-negative function for $t \geq 0$.

(b) f is continuous on R and non-negative.

(c) a is positive on $[0, \infty)$ such that $a(t) > a_0 > 0$ and $a'(t) \leq 0$ on $[0, \infty)$.

(d) g is continuous on R such that $G(x) = \int^x g(s)ds \to \infty$ as $O\left(|x|^{1+\delta}\right)$ where $\delta > 0$, and

(e) e is continuous on $[0, \infty)$ and $\int_0^x |e(s)|ds < \infty$.

Notice that we do not require that $xg(x) > 0$ for all $x \neq 0$. If the above conditions hold, all solutions as well as their derivatives are bounded as $t \to \infty$.

We now begin our analysis. First, using standard existence theory, we may conclude that the solutions to (1) are local [1, Chapter 6]. If we can show that the solutions remain bounded then we may conclude global existence of all solutions [1, pp.384-396].

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In order to see this, first multiply equation (1) by $x'(t)$ and then integrate from 0 to t where we integrate by parts the third term on the LHS of (1) obtaining

$$\begin{aligned}
&\frac{1}{2}\left(x'(t)\right)^2+\int_0^t c(s)f(x(s))\left(x'(s)\right)^2 ds+a(t)G(x(t)-\int_0^t a'(s)G(x(s))ds \\
=\ &\int{}_0^t e(s)x'(s)ds+\frac{1}{2}\left(x'(0)\right)^2+a(0)G(x(0)). \qquad (2)
\end{aligned}$$

Using the fact that e is absolutely integrable we see that

$$\begin{aligned}
&\frac{1}{2}\left(x'(t)\right)^2+\int_0^t c(s)f(x(s))\left(x'(s)\right)^2 ds+a(t)G(x(t)-\int_0^t a'(s)G(x(s))ds \\
\leq\ &\int_0^t |e(s)|x'(s)ds+\frac{1}{2}\left(x'(0)\right)^2+a(0)G(x(0)). \qquad (3)
\end{aligned}$$

Applying the mean value theorem for integrals to the term $\int_0^t |e(s)|x'(s)ds$ transforms equation (3) into

$$\begin{aligned}
&\frac{1}{2}\left(x'(t)\right)^2+\int_0^t c(s)f(x(s))\left(x'(s)\right)^2 ds+a(t)G(x(t))-\int_0^t a'(s)G(x(s))ds \\
\leq\ &x'(t^*)\int_0^\infty |e(s)|\,ds+\frac{1}{2}\left(x'(0)\right)^2+a(0)G(x(0)) \qquad (4)
\end{aligned}$$

where $0<t^*<t$. Should $|x|$ or $|x'|$ become unbounded, then, by our hypotheses, the LHS approaches ∞ as $O\left(|x|^{1+\delta}\right)$ and $O\left(|x'|^2\right)$ while the RHS approaches infinity as $O\left(|x'|\right)$. Since this is impossible, we have that both $|x|$ and $|x'|$ must stay bounded. This first result we will call Theorem I. Next we state Theorem II. The only difference in our hypothesis is $a'(t)\geq 0$. Also, the derivatives are not guaranteed to be bounded.

THEOREM II. The hypotheses are identical to Theorem I except that condition (c) changes to $a'(t)\geq 0$. Under those conditions, all solutions to (1) are bounded. If $a(t)$ is bounded from above by a positive constant A_0, then the derivatives, too, are bounded.

PROOF. In this case we multiply (1) by $x'(t)/a(t)$ and integrate from 0 to t where we integrate by parts only the first term on the LHS of (1) and proceeding as before we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\left(x'(t)\right)^2}{a(t)}+\frac{1}{2}\int_0^t \frac{x'(s)}{\left(a'(s)\right)^2}ds+\int_0^t \frac{c(s)f(x(s))\left(x'(s)\right)^2}{a(s)}ds+G(x(t)) \\
\leq\ &\frac{1}{2}\frac{\left(x'(0)\right)^2}{a(0)}+G(x(0))+x'(t^*)\int_0^t \frac{|e(s)|}{a(s)}ds. \qquad (5)
\end{aligned}$$

Again, notice that should either $|x|$ or $|x'|$ becomes infinite, the LHS of (5) would approach ∞ faster than the RHS of (5), so the solutions must remain bounded. Should $a(t)$ be bounded from above by some constant A_0, the derivatives, too, are bounded.

We now turn our attention to the case when e is bounded with a derivative of fixed sign. Statement and proof now follow.

THEOREM III. They hypotheses are the same as Theorem I except for condition (e). e is a bounded function with a derivative of fixed sign. In such cases, all solutions are bounded along with their derivatives.

PROOF. We first proceed as we did in Theorem I obtaining equation (2). However, we then integrate by parts the term $\int_0^t e(s)x'(s)ds$ obtaining

$$\begin{aligned} & \frac{1}{2}\left(x'(t)\right)^2 + \int_0^t c(s)f(x(s))\left(x'(s)\right)^2 ds + a(t)G(x(t)) - \int_0^t a'(s)G(x(s))ds \\ = \quad & e(t)x(t) - e(0)x(0) - \int{}_0^t e'(s)x(s)ds + \frac{1}{2}\left(x'(0)\right)^2 + a(0)G(x(0)). \qquad (6) \end{aligned}$$

We now apply the mean value theorem for integrals to the term $\int_0^t e'(s)x(s)ds$ which transform (6) into

$$\begin{aligned} & \frac{1}{2}\left(x'(t)\right)^2 + \int{}_0^t c(s)f(x(s))\left(x'(s)\right)^2 ds + a(t)G(x(t) - \int_0^t a'(s)G(x(s))ds \\ = \quad & x(t)e(t) - x(0)e(0) - x(t^*)(e(t) - e(0)) + \frac{1}{2}\left(x'(0)\right)^2 + a(0)G(x(0)), \qquad (7) \end{aligned}$$

where $0 < t^* < t$. Arguing as before, both $|x|$ and $|x'|$ must remain bounded. Otherwise, the LHS of (7) again becomes infinite faster than the RHS which is impossible. We now state our final theorem.

THEOREM IV. The hypothesis are the same as Theorem III except that $a'(t) \geq 0$ on $[0, \infty)$ and $a(t)e'(t) - a'(t)e(t) \neq 0$ on $[0, \infty)$, then all solutions are bounded as $t \to \infty$. Further if $a'(t)$ is bounded from above by some constant A_0, then the derivatives, too, are bounded.

PROOF. We proceed as in Theorem II to obtain,

$$\begin{aligned} & \frac{1}{2}\frac{\left(x'(t)\right)^2}{a(t)} + \frac{1}{2}\int_0^t \frac{x'(s)}{\left(a'(s)\right)^2}ds + \int_0^t \frac{c(s)f(x(s))\left(x'(s)\right)^2}{a(s)}ds + G(x(t)) \\ = \quad & \frac{1}{2}\frac{\left(x'(0)\right)^2}{a(0)} + G(x(0)) + \int_0^t \frac{x'(s)e(s)}{a(s)}ds. \qquad (8) \end{aligned}$$

Integrating by parts the last term of (8) we see that

$$\begin{aligned} & \frac{1}{2}\frac{\left(x'(t)\right)^2}{a(t)} + \frac{1}{2}\int_0^t \frac{x'(s)}{\left(a'(s)\right)^2}ds + \int_0^t \frac{c(s)f(x(s))\left(x'(s)\right)^2}{a(s)}ds + G(x(t)) \\ = \quad & \frac{1}{2}\frac{\left(x'(0)\right)^2}{a(0)} + G(x(0)) + \frac{x(t)e(t)}{a(t)} - \frac{x(0)e(0)}{a(0)} - \int_0^t \left(x(s)\frac{e(s)}{a(s)}\right)' ds. \qquad (9) \end{aligned}$$

As before, we apply the mean value theorem for integrals to the last term of (9) obtaining,

$$\begin{aligned} & \frac{1}{2}\frac{\left(x'(t)\right)^2}{a(t)} + \frac{1}{2}\int_0^t \frac{x'(s)}{\left(a'(s)\right)^2}ds + \int_0^t \frac{c(s)f(x(s))\left(x'(s)\right)^2}{a(s)}ds + G(x(t)) \\ = \quad & \frac{1}{2}\frac{\left(x'(0)\right)^2}{a(0)} + G(x(0)) + \frac{x(t)e(t)}{a(t)} - \frac{x(0)e(0)}{a(0)} - x(t^*)\int_0^t \left(\frac{e(s)}{a(s)}\right)' ds. \qquad (10) \end{aligned}$$

where $0 < t^* < t$. Simplifying (10) by integrating the last term yields finally,

$$\begin{aligned} & \frac{1}{2}\frac{(x'(t))^2}{a(t)} + \frac{1}{2}\int_0^t \frac{x'(s)}{(a'(s))^2}ds + \int_0^t \frac{c(s)f(x(s))\,(x'(s))^2}{a(s)}ds + G(x(t)) \\ = \quad & \frac{1}{2}\frac{x'(0)^2}{a(0)} + G(x(0)) + \frac{x(t)e(t)}{a(t)} - \frac{x(0)e(0)}{a(0)} - x(t^*)\left(\frac{e(t)}{a(t)} - \frac{e(0)}{a(0)}\right). \qquad (11) \end{aligned}$$

Equation (11) clearly show that $|x|$ remains bounded. Otherwise, the LHS of (11) approaches ∞ faster than the RHS which is a contradiction. Moreover, if $A_0 \geq a(t)$, then the derivatives, too, stay bounded for the same reason.

Under somewhat more restrictive condition, we can show that all solutions to (1) are in $L^p[0,\infty)$. Specifically, we assume e is an element of $L^1[0,\infty)$, $c(t) > c_0 > 0$, $c'(t) \leq 0$ and $f(x) > f_0 > 0$, $xg(x) > 0$ for $x \neq 0$, and $xg(x) = K\,|x|^p$ for $p \geq 2$ and K a positive constant. We immediately see that $|x'|$ is square integrable from (2). Next, multiply (1) by x and integrate the first term by parts getting

$$\begin{aligned} & x(t)x'(t) - \int{}_0^t x'(s)^2 ds + \int_0^t c(s)f(x(s))x(s)x'(s)ds + \int_0^t a(s)g(x(s))x(s)ds \\ = \quad & \int_0^t e(s)x(s)ds + x(0)x'(0). \qquad (12) \end{aligned}$$

Next, define $F(x) = \int^x f(u)udu$ and integrate the third term of (12) by parts so that (12) becomes,

$$\begin{aligned} & x(t)x'(t) - \int_0^t x'(s)^2 ds + c(t)F(x(t)) - \int_0^t c'(s)F(x(s))ds \\ & + \int_0^t a(s)g(x(s))x(s)ds \\ = \quad & \int_0^t e(s)x(s)ds + x(0)x'(0) + c(0)F(x(0)). \qquad (13) \end{aligned}$$

From (13), since all terms on the LHS are positive and all terms on the RHS are bounded, we may conclude that $|x|$ is indeed an element of $L^p[0,\infty)$

As an example, consider the following equation

$$x'' + kx' + tp(x) = C \qquad (14)$$

where $p(x)$ is a polynomial of odd degree $2n+1$ and k and C are constants. From our above remarks, all solutions are bounded.

Next, consider the equation,

$$x'' + (t + \sin(t))x' + tp(x) = C \qquad (15)$$

where the constant has been replaced by $t + \sin t$. All solutions to (15) are bounded.

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