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Several Identities And Inequalities Involving Jordan Closed Curves*

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Abstract

In this paper, we define an outer adjoint curve and outer adjoint region of a Jordan closed curve. Under proper hypotheses, we obtain several identities and inequalities involving the outer adjoint curves and the outer adjoint regions of Jordan closed curves.

1 Introduction

We first consider the following problem: Let there be two simple closed curves in the plane which are parallel with distance d apart from each other. Let C denote the 'inner curve' and let $\Omega(C;d)$ denote the region bounded between these two curves. What is the length of the 'outer curve' and what is the area of $\Omega(C;d)$?

Such a problem is meaningful since the two simple closed curves can be used to represent a railway track, or the two sides of a ditch or moat around a city.

In this paper, we will describe our problem in more precise mathematical terms and solve the corresponding problem.

Let $C = C(AB) : z = z(t)$ $(a \leqslant t \leqslant b)$ be a continuous curve in a complex plane, where $A = z(a)$ and $B = z(b)$. The point A is called the initial point and B the terminal point of C. The continuous curve C is said to be smooth at the point $z(t_0)$ where $t_0 \in [a, b]$ if there exists a tangent line at this point. The curve C is smooth if it is smooth at each point of C, and piecewise smooth if it is smooth at all points of C except for a finite number of points.

If there exist t_1, t_2 such that $a \leqslant t_1 \leqslant b$, $a < t_2 < b$, $t_1 \neq t_2$ and $z(t_1) = z(t_2)$, then $z(t_1)$ is called a coincident point of C. A continuous curve without any coincident point is a Jordan curve. If a Jordan curve C satisfies $A = B$, then it is a Jordan closed curve. $|C(AB)|$, $|C|$ and $|AB|$ denote respectively the length of $C(AB)$, the length of the Jordan closed curve C and the length of the line segment AB. $D(C)$ denotes the region enclosed by the Jordan closed curve C and $|D(C)|$ denotes the area of $D(C)$.

For a Jordan closed curve, we have the following theorem.

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THEOREM. An arbitrary Jordan closed curve must divide a plane into two parts, where one part is bounded and another is unbounded. The bounded part is called the interior and the other the exterior of the Jordan closed curve.

Traveling along a Jordan closed curve C, if the interior of C on the left (right) hand, we call the traveling direction is in the positive (respectively negative) direction of C.

DEFINITION 1. Let $C = C(AB)$ be a smooth Jordan curve and $P \in C$. Let l_P be the tangent line of C and n_P the normal line of C at the point P. Suppose that $Q \in n_P$ and $|PQ| = d > 0$ (d is a constant), then we say that the trail of Q, written as $C' = C'(A'B')$, is an adjoint curve of C as P moves continuously from A to B along C. The trail of the line segment PQ, written as $\Omega(C; d)$, is called adjoint region of C with width d. In particular, if $A = B$ and the starting point Q is in the outside (interior) of C, C' is called the outer (respectively inner) adjoint curve of C and $\Omega(C; d)$ is outer (respectively inner) adjoint region of C with width d.

REMARK 1. The trail and set are two different concepts. If a moving point goes through the same point twice, the point ought to be calculated twice in the trail.

For a piecewise smooth Jordan curve $C = C(AB)$, we may define an adjoint curve and an adjoint region as follows. Let $C(AB)$ be partitioned into N parts with equal lengths by $A_0 = A, A_1, ..., A_{i-1}, A_i, A_{i+1}, ..., A_N = B$. Let $\odot(O_i, r)$ be a circle with center O_i and (sufficiently small) radius r $(0 < r < d)$ such that it is tangent to the rays $A_{i-1}A_i$ and A_iA_{i+1}. Set the tangent points $T_i \in A_{i-1}A_i$ and $T_i^* \in A_iA_{i+1}$. Hence the broken line $C_N = AA_1 \cdots A_{i-1}A_iA_{i+1} \cdots B$ can be extended to a smooth curve

$$C_{N,r} = C_{N,r}(A\widetilde{T_1T_1^*}A_2 \cdots A_{i-1}\widetilde{T_iT_i^*}A_{i+1} \cdots A_{N-2}\widetilde{T_{N-1}T_{N-1}^*}B),$$

where the curve $\widetilde{T_iT_i^*}$ denotes a circular arc on $\odot(O_i, r)$ $(i = 1, 2, ..., N-1)$. Hence we have the following definition.

DEFINITION 2. Let $C = C(AB)$ be a piecewise smooth Jordan curve and

$$C' = C'(A'B') := \lim_{N\to\infty} \lim_{r\to 0} C'_{N,r}(A'B')$$

and

$$\Omega(C; d) := \lim_{N\to\infty} \lim_{r\to 0} \Omega(C_{N,r}; d).$$

Then C' is called an adjoint curve of C and $\Omega(C; d)$ is an adjoint region of C with width d.

REMARK 2. Since $C = C(AB)$ is a piecewise smooth curve in the plane, we see that

$$\lim_{N\to\infty} \lim_{r\to 0} C'_{N,r}(A'B') \text{ and } \lim_{N\to\infty} \lim_{r\to 0} \Omega(C_{N,r}; d)$$

exist.

DEFINITION 3. Let $C = C(AB)$ be a smooth Jordan curve and $P \in C$. Let l_P be the tangent line of C and n_P the normal line of C at P. Suppose that $Q \in n_P$ and $|PQ| = d > 0$. When P moves continuously from A to B along C, Q moves continuously along the adjoint curve C' of C.

- If $\forall \varepsilon > 0 \Rightarrow \exists P_1 \in C$ such that $0 < |C(PP_1)| < \varepsilon$ and $PQ \cap P_1Q_1 = \emptyset$, then Q is called a positive point of C'.
- If $\forall \varepsilon > 0 \Rightarrow \exists P_1 \in C$ such that $0 < |C(PP_1)| < \varepsilon$ and $PQ \cap P_1Q_1 \neq \emptyset$, then Q is called a negative point of C'.
- If Q is a positive and a negative point of C', then Q is also called a zero point of C'.

The above definition is motivated by observing the movement of a cart with two parallel wheels. When a street corner is encountered, we may either hold one wheel, say the left wheel still, and then push the right wheel to make a left turn; or we may hold the right wheel still, pull back the left wheel a good distance and then push both wheels to make the same left turn. In the former case, the right wheel will trace out positive points; while in the latter case, the left wheel will trace out two zero points and negative points (see Figure 1).

REMARK 3. As P moves continuously from A to B along a smooth Jordan curve $C = C(AB)$, the trail of the positive (negative) point Q of C', written as C'_+ (respectively C'_-), is composed of some continuous segments (possibly with common segments) and the sum of their lengths is written as $\left|C'_+\right|$ (respectively $\left|C'_-\right|$); the trail C'_0 of the zero point is composed of some isolated points and the measure of C'_0 is $|C'_0| = 0$. The trail Ω_+ (Ω_-) of the positive (respectively negative) points in the line segment PQ is composed of some regions (which may intersect each other) and $|\Omega_+|$ (respectively $|\Omega_-|$) denotes the sum of areas of these regions. The trail Ω_0 of the zero point in the line segment PQ is also composed of some isolated points and the measure of Ω_0 is $|\Omega_0| = 0$.

DEFINITION 4. If $C = C(AB)$ is a smooth Jordan curve, we write

$$\overline{|C'(A'B')|} := \left|C'_+\right| - \left|C'_-\right| \quad \text{and} \quad \overline{|\Omega(C;d)|} := |\Omega_+| - |\Omega_-|\,;$$

and if $C = C(AB)$ is a piecewise smooth Jordan curve and $|C(AB)|$ exists, we write

$$\overline{|C'(A'B')|} := \lim_{N\to\infty}\lim_{r\to 0}\overline{\left|C'_{N,r}(A'B')\right|} \text{ and } \overline{|\Omega(C;d)|} := \lim_{N\to\infty}\lim_{r\to 0}\overline{|\Omega(C_{N,r};d)|}.$$

$\overline{|C'(A'B')|}$ is called the algebraic length of $C = C'(A'B')$ and $\overline{|\Omega(C;d)|}$ is called the algebraic area of $\Omega(C;d)$ (see Figure 1).

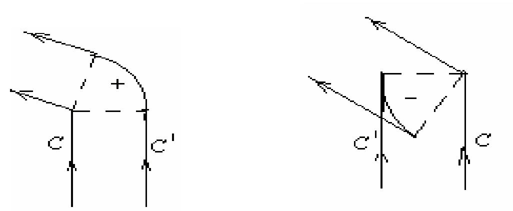

Figure 1

DEFINITION 5. Given two rays OA and OB in the plane, the angle $\overline{\angle AOB}$ generated by revolving counterclockwise OA to OB about O is called the oriented angle from OA to OB and satisfies $\overline{\angle AOB} \in [0, 2\pi], \overline{\angle AOB} + \overline{\angle BOA} = 2\pi$.

2 The Main Result

We have the following main result.

THEOREM 1. Let C be a smooth or piecewise smooth Jordan closed curve. Let C' be the outer adjoint curve of C and $\Omega(C; d)$ be the outer adjoint region of C. Then we have

$$|C'| \geqslant \overline{|C'|} = |C| + 2\pi d \tag{1}$$

and

$$|\Omega(C; d)| \leqslant \overline{|\Omega(C; d)|} = d(|C| + \pi d). \tag{2}$$

The sufficient condition for equalities to hold in (1) and (2) is that C' is a Jordan closed curve.

PROOF. Since C is a Jordan closed curve and according to the above definitions, we may suppose that $C = C_N = A_1 \cdots A_{i-1}A_iA_{i+1} \cdots A_NA_1$ is a polygon with N sides and its positive direction is: $A_1 \to A_2 \to \cdots \to A_N \to A_1$. We define $A_i = A_j \Leftrightarrow i \equiv j($ mod $N), \forall i, j \in Z$, where Z is the set of integers (see [1]). Then we have

$$\sum_{i=1}^{N}(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) = 2\pi. \tag{3}$$

If C is a convex polygon with N sides, we have

$$C'_- = \emptyset,\ |C'_-| = 0,\ \Omega_- = \emptyset,\ |\Omega_-| = 0,$$

and

$$0 < \overline{\angle A_{i+1}A_iA_{i-1}} \leqslant \pi\ (i = 1, 2, ..., N).$$

Moreover, we know that C'_N is composed of N rectangles and N sector arcs of radius d with centers at the vertexes of C (see Figure 2). By the definitions of algebraic length and algebraic area, the equality (3) and $\sum_{i=1}^{N} |A_{i-1}A_i| = |C|$, we get

$$|C'| = \overline{|C'|} = \sum_{i=1}^{N} |A_{i-1}A_i| + \sum_{i=1}^{N} d(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) = |C| + 2\pi d$$

and

$$|\Omega(C; d)| = \overline{|\Omega(C; d)|} = \sum_{i=1}^{N} d|A_{i-1}A_i| + \sum_{i=1}^{N} \frac{1}{2}d^2(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) = d(|C| + \pi d).$$

If C is a non-convex polygon with N sides, then

$$\exists\overline{\angle A_{i+1}A_iA_{i-1}} : \pi \leqslant \overline{\angle A_{i+1}A_iA_{i-1}} < 2\pi, \tag{4}$$

C'_N is also composed of some rectangles and some sector arcs of radius d with centers at the vertexes of C (see Figure 2). The points in rectangles are positive points and the points in sectors are negative points. Then the central angle, arc length and area of sectors are respectively

$$\pi-\overline{\angle A_{i-1}A_iA_{i+1}} = -(\pi-\overline{\angle A_{i+1}A_iA_{i-1}}), -d(\pi-\overline{\angle A_{i+1}A_iA_{i-1}}), -\frac{1}{2}d^2(\pi-\overline{\angle A_{i+1}A_iA_{i-1}}).$$

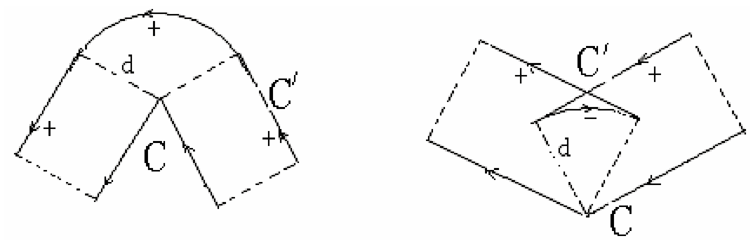

Figure 2

Let

$$I_- = \{i|\pi \leqslant \overline{\angle A_{i+1}A_iA_{i-1}} < 2\pi, 1 \leqslant i \leqslant N\}$$

and

$$I_+ = \{1, 2, ..., N\}\backslash I_-.$$

By the definitions of algebraic length and algebraic area, the equality (3) and $\sum_{i=1}^N |A_{i-1}A_i| = |C|$, we obtain that

$$\begin{aligned}\overline{|C'|} &= \sum_{i=1}^N |A_{i-1}A_i| + \sum_{i\in I_+} d(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) \\ &\quad - \sum_{i\in I_-} d\left[-(\pi - \overline{\angle A_{i+1}A_iA_{i-1}})\right] \\ &= |C| + 2\pi d\end{aligned}$$

and

$$\begin{aligned}\overline{|\Omega(C;d)|} &= \sum_{i=1}^N d|A_{i-1}A_i| + \sum_{i\in I_+} \frac{1}{2}d^2(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) \\ &\quad - \sum_{i\in I_-} \frac{1}{2}d^2\left[-(\pi - \overline{\angle A_{i+1}A_iA_{i-1}})\right] \\ &= d|C| + \frac{1}{2}d^2\sum_{i=1}^N \left(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}\right) \\ &= d(|C| + \pi d).\end{aligned}$$

It follows from the definitions of algebraic length and algebraic area that

$$|C'| = |C'_+| + |C'_-| \geqslant |C'_+| - |C'_-| = \overline{|C'|} = |C| + 2\pi d$$

and

$$\begin{aligned} |\Omega(C;d)| &= \sum_{i=1}^{N} d|A_{i-1}A_i| + \sum_{i\in I_+} \frac{1}{2} d^2(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) \\ &\quad - \sum_{i\in I_-} \left|(A_{i+1}A_iA'_iA'_{i+1}) \cap (A_iA_{i-1}A'_{i-1}A'_i)\right| \\ &\leqslant \sum_{i=1}^{N} d|A_{i-1}A_i| + \sum_{i\in I_+} \frac{1}{2} d^2(\pi - \overline{\angle A_{i+1}A_iA_{i-1}}) \\ &\quad - \sum_{i\in I_-} \frac{1}{2} d^2 \left[-(\pi - \overline{\angle A_{i+1}A_iA_{i-1}})\right] \\ &= d(|C| + \pi d), \end{aligned}$$

where $(A_{i+1}A_iA'_iA'_{i+1})$ and $(A_iA_{i-1}A'_{i-1}A'_i)$ are rectangles and

$$(A_{i+1}A_iA'_iA'_{i+1}) \cap (A_iA_{i-1}A'_{i-1}A'_i)$$

is a common region of them. Hence (1) and (2) are proved. In particular, if C' is a Jordan closed curve, then

$$C'_- = \emptyset, |C'_-| = 0, \Omega_- = \emptyset, |\Omega_-| = 0,$$

$$|C'| = |C'_+| + |C'_-| = |C'_+| - |C'_-| = \overline{|C'|} = |C| + 2\pi d$$

and

$$\Omega(C;d) = |\Omega'_+| = |\Omega'_+| - |\Omega'_-| = \overline{\Omega(C,d)} = d(|C| + \pi d).$$

Hence equalities in (1) and (2) hold. This completes the proof.

REMARK 4. It follows from Theorem 1 that the original problem in the Introduction is solved.

From Theorem 1 and the isoperimetric inequality ($4\pi S \leqslant |C|^2$, see [2-7]), we obtain the following.

COROLLARY 1. Suppose that C and C' are two smooth or piecewise smooth Jordan closed curves, C' is the outer adjoint curve of C, $\Omega(C;d)$ is the outer adjoint region of C, $|C|$ exists and $S := |D(C)|$, then we have

$$|C'| \geqslant 2\sqrt{\pi S} + 2\pi d, \tag{5}$$

$$|\Omega(C;d)| \geqslant d(2\sqrt{\pi S} + \pi d), \tag{6}$$

where equalities hold if and only if C is a circle.

COROLLARY 2. Suppose the curve $\underline{C(AB)} + BA$ is a smooth or piecewise smooth Jordan closed curve, $\overline{AA^*}$ and $\overline{BB^*}$ are the two tangent lines at the points A and B

respectively. Write $\overline{\angle A^*AB} = \alpha$ and $\overline{\angle ABB^*} = \beta$ (see Figure 3). The outer adjoint curve $C'(A'B') + B'A'$ of $C(AB) + BA$ is also a Jordan closed curve; $\Omega(C(AB); d)$ is the outer adjoint region of $C(AB) + BA$ with width d. Then we have

$$|C'(A'B')| = |C(AB)| + (\alpha + \beta)d \tag{7}$$

and

$$|\Omega(C(AB); d)| = d\left(|C(AB)| + \frac{\alpha + \beta}{2}d\right). \tag{8}$$

PROOF. By Theorem 1, we have (see Figure 3)

$$|C'(A'B')| + (\pi - \alpha)d + |AB| + (\pi - \beta)d = |C(AB)| + |AB| + 2\pi d$$

which implies

$$|C'(A'B')| = |C(AB)| + (\alpha + \beta)d;$$

and

$$|\Omega(C(AB); d)| + \frac{1}{2}(\pi - \alpha)d^2 + |AB|d + \frac{1}{2}(\pi - \beta)d^2 = d\left[(|C(AB)| + |AB|) + \pi d\right]$$

which implies

$$|\Omega(C(AB); d)| = d\left(|C(AB)| + \frac{\alpha + \beta}{2}d\right).$$

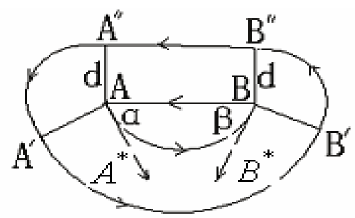

Figure 3

This ends the proof.

EXAMPLE 1. Let C be a square of side length 12 and $d = 1$. Then C and C' are Jordan closed curves. By Theorem 1, we know

$$|C'| = \overline{|C'|} = |C| + 2\pi d = 48 + 2\pi$$

and

$$\Omega(C; d) = \overline{\Omega(C; d)} = d(|C| + \pi d) = 48 + \pi.$$

EXAMPLE 2. Let C be composed of three sides of a square of side length 12 and three semicircles of radius 2, where the diameters of three semicircles are on the fourth side of the square, the semicircle in the middle is convex to the interior of the square and another two semicircles are convex to the outside of the square, $d = 1$ (see Figure 4). Then C and C' are Jordan closed curves.

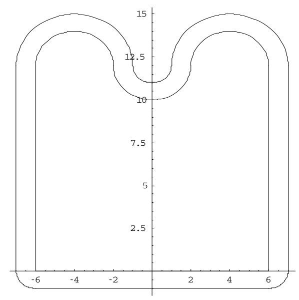

Figure 4

By Theorem 1, we get

$$|C'| = \overline{|C'|} = |C| + 2\pi d = 36 + 8\pi$$

and

$$\Omega(C; d) = \overline{\Omega(C; d)} = d(|C| + \pi d) = 36 + 7\pi.$$

3 Applications

An important application of our main result is the following.

THEOREM 2. Let $D_1, D_2, ..., D_n$ $(D_1 \subset D_2 \subset \cdots \subset D_n, n \geqslant 3)$ be n simply connected regions in the same plane, $C_i := \partial(D_i)$ be the boundary curve of D_i, $S_i := |D(C_i)| = |D_i|$ and the real number $d > 0$, $i = 1, 2, ..., n$. If (i) $C_1, C_2, ..., C_n$ are smooth Jordan closed curves; (ii) C_{i+1} is the outer adjoint curve of C_i, $i = 1, 2, ..., n-1$; and (iii) for the real number $p_i > 0, i = 1, 2, ..., n$, $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n ip_i = m$, $1 < m < n$, then we have

$$\sum_{i=1}^{n} p_i \sqrt{S_i} \leqslant \sqrt{S_m} < \sqrt{\sum_{i=1}^{n} p_i S_i} < \sum_{i=1}^{n} \sqrt{p_i S_i}, \tag{9}$$

where $S_m := S_1 + (m-1)[|C_1| + (m-1)\pi]$. Equality in (9) holds if and only if D_1 is a disc.

PROOF. Set $d = 1$. By Theorem 1, we have

$$|C_{k+1}| = |C_k| + 2\pi \ (k = 1, 2, ..., n-1),$$

$$|C_k| = |C_1| + 2(k-1)\pi \ (k = 1, 2, ..., n),$$

$$S_{k+1} - S_k = |C_k| + \pi = |C_1| + (2k-1)\pi$$

for $k = 1, 2, ..., n-1$, and

$$\begin{aligned} S_k &= S_1 + \sum_{j=1}^{k-1} [|C_1| + (2j-1)\pi] \\ &= S_1 + (k-1)[|C_1| + (k-1)\pi] \end{aligned}$$

for $k = 1, 2, ..., n$. Let the function $\varphi : [0, +\infty) \to R$ be defined by

$$\varphi(t) = \sqrt{S_1 + t(|C_1| + \pi t)}$$

and the function $\psi : [0, +\infty) \to R$ by

$$\psi(t) = S_1 + t(|C_1| + \pi t).$$

Then the first and second inequalities in (9) become respectively

$$\sum_{i=0}^{n-1} p_{i+1}\varphi(i) \leqslant \varphi\left(\sum_{i=0}^{n-1} ip_{i+1}\right) \tag{10}$$

and

$$\sum_{i=0}^{n-1} p_{i+1}\psi(i) > \psi\left(\sum_{i=0}^{n-1} ip_{i+1}\right). \tag{11}$$

Differentiating $\varphi(t)$ and $\psi(t)$ with respect to t and by the isoperimetric inequality (see [2-7]), we obtain

$$\varphi'(t) = \frac{|C_1| + 2\pi t}{2\sqrt{S_1 + t(|C_1| + \pi t)}},$$

$$\begin{aligned}
\varphi''(t) &= \frac{2\pi\sqrt{S_1 + t(|C_1| + \pi t)} - \frac{1}{2}\frac{(|C_1|+2\pi t)^2}{\sqrt{S_1+t(|C_1|+\pi t)}}}{2[S_1 + t(|C_1| + \pi t)]} \\
&= \frac{4\pi[S_1 + t(|C_1| + \pi t)] - (|C_1| + 2\pi t)^2}{4[S_1 + t(|C_1| + \pi t)]\sqrt{S_1 + t(|C_1| + \pi t)}} \\
&= \frac{4\pi S_1 - |C_1|^2}{4[S_1 + t(|C_1| + \pi t)]\sqrt{S_1 + t(|C_1| + \pi t)}} \\
&\leqslant 0,
\end{aligned} \tag{12}$$

and

$$\psi''(t) = 2\pi > 0.$$

Consequently, $\varphi : [0, +\infty) \to R$ is a concave function and $\psi : [0, +\infty) \to R$ is a convex function. It follows from the Jensen inequalities of convex functions (see [8-9]) that (10) and (11) hold, thus, the first and second inequalities in (9) hold. And since $0, 1, 2, ..., n-1$ are not equal, equality in (9) occurs if and only if $\varphi''(t) \equiv 0 (\forall t \geqslant 0) \Leftrightarrow 4\pi S_1 - |C_1|^2 = 0 \Leftrightarrow D_1$ is a disc. Certainly, the last inequality in (9) holds. This completes the proof.

By the power means inequality (see [10-13]):

$$\sum_{i=1}^{n} p_i x_i^r \leq \left(\sum_{i=1}^{n} p_i x_i\right)^r, p_i > 0, x_i \geq 0, i = 1, 2, ..., n, \sum_{i=1}^{n} p_i = 1, 0 < r \leq 1,$$

and Jensen inequality:

$$\sum_{i=1}^{n} x_i^r \geq \left(\sum_{i=1}^{n} x_i\right)^r, x_i \geq 0, i = 1, 2, ..., n, 0 < r \leq 1,$$

we get the following corollary.

COROLLARY 3. Under the conditions of Theorem 2, for a real number $0 < r \leqslant \frac{1}{2}$, we have

$$\sum_{i=1}^{n} p_i S_i^r \leqslant S_m^r < \sum_{i=1}^{n} (p_i S_i)^r. \tag{13}$$

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