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The Arrow Impossibility Theorem Of Social Choice Theory In An Infinite Society And Limited Principle Of Omniscience*

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Abstract

This paper is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. We examine the Arrow impossibility theorem ([1]) in a society with an infinite number of individuals (infinite society). We will show that the theorem that there exists a dictator or there exists no dictator for any binary social choice rule satisfying transitivity, Pareto principle and independence of irrelevant alternatives in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

1 Introduction

This paper is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics[1]. Arrow's impossibility theorem ([1]) shows that, with a finite number of individuals, for any social welfare function (transitive binary social choice rule) which satisfies Pareto principle and independence of irrelevant alternatives (IIA) there exists a dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter. On the other hand, [5], [6] and [7] show that, in a society with an infinite number of individuals (infinite society), there exists a social welfare function satisfying Pareto principle and IIA without dictator[2].

In this paper we will show that the theorem that there exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive.

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[1]In other papers, for example [13], we study the relationships of some other theorems of social choice theory with LPO.

[2][14] is a recent book that discusses social choice problems in an infinite society.

The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs[3]. This is done by showing that the statement implies the omniscience principle. The strongest omniscience principle is the law of excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

LPO (Limited principle of omniscience) Given a binary sequence a_n, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, either $a_n = 0$ for all n or $a_n = 1$ for some n.

In the next section we present the framework of this paper and some preliminary results. In Section 3 we will show the following results.

1. There exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 1).

2. Theorem 1 is equivalent to LPO (Theorem 2).

A decisive set is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the society prefers x to y regardless of the preferences of other individuals.

2 The framework and preliminary results

There are more than two (finite or infinite) alternatives and a countably infinite number of individuals. The set of individuals is denoted by N. The set of alternatives is denoted by A. N and A are discrete sets[4]. For each pair of elements i, j of N we have $i = j$ or $i \neq j$, and for each pair of elements x, y of A we have $x = y$ or $x \neq y$. Each subset of N is detachable. Thus, for each individual i of N and each subset I of N we have $i \in I$ or $i \notin I$. The alternatives are represented by x, y, z, w and so on. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Individual preferences over the alternatives are transitive weak orders, and they are characterized constructively according to [2]. About given three alternatives x, y and z individual i's preference satisfies the following properties.

1. If $x \succ_i y$, then $\neg(y \succ_i x)$.

2. If $x \succ_i y$, then for each $z \in A$ either $x \succ_i z$ or $z \succ_i y$.

Preference-indifference relation $\succsim_i$ and indifference relation $\sim_i$ are defined by

- $x \succsim_i y$ if and only if $\forall z \in A(y \succ_i z \Rightarrow x \succ_i z)$,
- $x \sim_i y$ if and only if $x \succsim_i y$ and $y \succsim_i x$.

[3]About omniscience principles we referred to [3], [4], [8] and [9].
[4]About details of the concepts of discrete set and detachable set, see [3].

Then, the following results are derived.

- $\neg(x \succ_i x)$.
- $x \succ_i y$ entails $x \succsim_i y$.
- The relations $\succ_i$, $\succsim_i$ are transitive, and $x \succsim_i y \succ_i z$ entails $x \succ_i z$.
- $x \succsim_i y$ if and only if $\neg(y \succ_i x)$.

As demonstrated by [2] we can not prove constructively that $x \succ_i y$ if and only if $\neg(y \succsim_i x)$.

A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (\succ_1, \succ_2, \cdots))$, $\mathbf{p}'(= (\succ_1', \succ_2', \cdots))$ and so on.

We consider a binary social choice rule which determines a social preference corresponding to each profile. Social preferences are defined similarly to individual preferences. We denote $x \succ y$ when the society strictly prefers x to y. The social preference is denoted by $\succ$ at $\mathbf{p}$, by $\succ'$ at $\mathbf{p}'$ and so on, and it satisfies the following conditions.

1. P1: If $x \succ y$, then $\neg(y \succ x)$.
2. P2: If $x \succ y$, then for each $z \in A$ either $x \succ z$ or $z \succ y$.

$x \succsim y$ and $x \sim y$ are defined as follows.

- $x \succsim y$ if and only if $\forall z \in A(y \succ z \Rightarrow x \succ z)$,
- $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$.

Then, the following results are derived.

- $\neg(x \succ x)$
- $x \succ y$ entails $x \succsim y$.
- The relations $\succ$, $\succsim$ are transitive, and $x \succsim y \succ z$ entails $x \succ z$.
- $x \succsim y$ if and only if $\neg(y \succ x)$.

Social preferences are further required to satisfy *Pareto principle* and *independence of irrelevant alternatives (IIA)*. The meanings of these conditions are as follows.

Pareto principle When all individuals prefer x to y, the society must prefer x to y.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x and y is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x and y.

A binary social choice rule which satisfies transitivity is called a *social welfare function*. Arrow's impossibility theorem ([1]) shows that, with a finite number of individuals, for any social welfare function satisfying Pareto principle and IIA there exists a dictator. In contrast [5], [6] and [7] show that when the number of individuals in the society is infinite, there exists a social welfare function satisfying Pareto principle and IIA without dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

According to definitions in [11] we define the following terms.

Almost decisiveness If, when all individuals in a (finite or infinite) group G prefer an alternative x to another alternative y, and other individuals (individuals in $N \setminus G$) prefer y to x, the society prefers x to y ($x \succ y$), then G is *almost decisive* for x against y.

Decisiveness If, when all individuals in a group G prefer x to y, the society prefers x to y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set If a group of individuals is decisive about every pair of alternatives, it is called a decisive set.

A decisive set may consist of one individual. If an individual is decisive about every pair of alternatives for a social welfare function, then he is a *dictator* of the social welfare function. Of course, there exists at most one dictator.

First about decisiveness we show the following lemma.

LEMMA 1. If a group of individuals G is almost decisive for an alternative x against another alternative y, then it is decisive about every pair of alternatives, that is, it is a decisive set.

PROOF. See Appendix.

The implications of this lemma are similar to those of Lemma 3*a in [11] and Dictator Lemma in [12]. Next we show the following lemma.

LEMMA 2. If G_1 and G_2 are decisive sets, then $G_1 \cap G_2$ is also a decisive set.

PROOF. Let x, y and z be given three alternatives, and consider the following profile.

1. Individuals in $G_1 \setminus G_2$ (denoted by i): $z \succ_i x \succ_i y$
2. Individuals in $G_2 \setminus G_1$ (denoted by j): $y \succ_j z \succ_j x$
3. Individuals in $G_1 \cap G_2$ (denoted by k): $x \succ_k y \succ_k z$
4. Other individuals (denoted by l): $z \succ_l y \succ_l x$

Since G_1 and G_2 are decisive sets, the social preference is $x \succ y$ and $y \succ z$. Then, by transitivity the social preference about x and z should be $x \succ z$. Only individuals in $G_1 \cap G_2$ prefer x to z, and all other individuals prefer z to x. Thus, $G_1 \cap G_2$ is almost decisive for x against z. Then, by Lemma 1 it is a decisive set.

Note that G_1 and G_2 can not be disjoint. Assume that G_1 and G_2 are disjoint. If individuals in G_1 prefer x to y, and individuals in G_2 prefer y to x, then neither G_1 nor G_2 can be a decisive set.

This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

3 Existence of social welfare function satisfying Pareto principle and IIA without dictator and LPO

Consider profiles such that one individual (denoted by i) prefers x to y to z, and all other individuals prefer z to x to y. Denote such a profile by $\mathbf{p}^i$. By Pareto principle the social preference about x and y is $x \succ y$. By the property of constructively defined social preference (P2) the social preference is $x \succ z$ or $z \succ y$. If it is $x \succ z$ at $\mathbf{p}^i$ for some i, then by IIA individual i is almost decisive for x against z, and by Lemma 1 he is a dictator. On the other hand, if the social preference is $z \succ y$ at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator. In this case by IIA, Lemma 1 and 2 all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain

THEOREM 1. For any social welfare function satisfying Pareto principle and IIA there exists a dictator or there exists no dictator, and in the latter case all co-finite sets are decisive sets.

But we can show the following theorem.

THEOREM 2. Theorem 1 is equivalent to LPO.

PROOF. Define a binary sequence (a_i) as follows.

$a_i = 1$ for $i \in \mathbb{N}$ if the social preference about x and z at $\mathbf{p}^i$ is $x \succ z$

$a_i = 0$ for $i \in \mathbb{N}$ if the social preference about y and z at $\mathbf{p}^i$ is $z \succ y$

The condition of LPO for this binary sequence is as follows.

LPO (Limited principle of omniscience)

$$a_i = 0 \text{ for all } i \in \mathbb{N} \text{ or } a_i = 1 \text{ for some } i \in \mathbb{N}$$

From the arguments before Theorem 1 it is clearly equivalent to Theorem 1.

Note $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i. Assume $x \succ z$ and $z \succ y$ at $\mathbf{p}^i$, and consider the following profile.

1. Individual i: $y \succ_i x \succ_i z$
2. Other individuals (denoted by j): $z \succ_j y \succ_j x$

By IIA the social preference is $x \succ z$ and $z \succ y$. Then, by transitivity the social preference about x and y must be $x \succ y$. It means $\neg(y \succ x)$. But by Pareto principle the social preference must be $y \succ x$. Therefore, $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$.

4 Concluding Remarks

We have examined the Arrow impossibility theorem of social choice theory in an infinite society, and have shown that the theorem that there exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA in an infinite society is equivalent to LPO (Limited principle of omniscience), and so it is non-constructive. The assumption of an infinite society seems to be unrealistic. But [10] presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

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Appendix: Proof of Lemma 1

1. Case 1: There are more than three alternatives.

   Let z and w be alternatives other than x and y, and consider the following profile.

   (a) Individuals in G (denoted by i): $z \succ_i x \succ_i y \succ_i w$.

   (b) Other individuals (denoted by j): $y \succ_j x$, $z \succ_j x$ and $y \succ_j w$. Their preferences about z and w are not specified.

   By Pareto principle the social preference is $z \succ x$ and $y \succ w$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $z \succ w$. This means that G is decisive for z against w. From this result we can show that G is decisive for x (or y) against w, for z against x (or y), for y against x, and for x against y. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.

2. Case 2: There are only three alternatives x, y and z.

   Consider the following profile.

   (a) Individuals in G (denoted by i): $x \succ_i y \succ_i z$.

   (b) Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

   By Pareto principle the social preference is $y \succ z$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $x \succ z$. This means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.

   (a) Individuals in G (denoted by i): $z \succ_i x \succ_i y$.

   (b) Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

   By similar procedures we can show that G is decisive for y against z, for z against x, for y against x, and for x against y.

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