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An Explicit Factorization Of Totally Positive Generalized Vandermonde Matrices Avoiding Schur Functions*

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Abstract

We consider a totally positive generalized Vandermonde matrix and obtain by induction its unique LU factorization avoiding Schur functions. As by-products, we get a recursive formula for the determinant and the inverse of a totally positive generalized Vandermonde matrix and express any Schur function in an explicit form.

1 Introduction

Let

$$G_{\{n;a_1,a_2,\cdots,a_n\}} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & \cdots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & \cdots & x_2^{a_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_n} \end{bmatrix}$$

be a totally positive (TP) generalized Vandermonde matrix [1, p.142], where $0 \le a_1 < a_2 < \cdots < a_n$ are integers and $0 < x_1 < x_2 < \cdots < x_n$. In connection with Schur functions we define the partition λ associated with G as the nonincreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) = (a_n - (n-1), a_{n-1} - (n-2), \cdots, a_1),$$

and get

$$G_{\{n;a_1,a_2,\cdots,a_n\}} = [x_i^{j-1+\lambda_{n-j+1}}]_{1\le i,j\le n},$$

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and Schur function associated to λ is defined as

$$s_\lambda(x_1, x_2, \cdots, x_n) = \frac{\det G_{\{n;a_1,a_2,\cdots,a_n\}}}{\det G_{\{n;0,1,\cdots,n-1\}}}.$$

In a recent paper [1], J. Demmel and P. Koev performed accurate and efficient matrix computations of $G_n := G_{\{n;a_1,a_2,\cdots,a_n\}}$. Their results, among others, are an explicit bidiagonal decompositions of G_n^{-1} and an LDU decomposition of G_n which involves Schur functions.

In a previous paper [2], using only mathematical induction, we succeeded to provide the unique LU factorization of a special case of generalized Vandermonde matrices, and this prompts us to do the same thing with G_n. Our main result presents an explicit LU factorization not involving Schur functions. As by-products, we calculate the determinant, the inverse of G_n and the Schur functions $s_\lambda(x_1, x_2, \cdots, x_n)$.

2 The LU factorization of G_n and applications

An LU factorization is to express a matrix as a product of a lower triangular matrix L and an upper triangular matrix U. If L is a lower triangular matrix with unit main diagonal and U is an upper triangular matrix, the LU factorization is unique. Our main aim in this note is to give the explicit unique LU factorization of G_n without involving Schur functions; Demmel and Koev provided in [1], Section 3, an explicit bidiagonal decomposition of G_n^{-1} which involves Schur functions. Our main result is the following theorem whose proof occupies almost the whole Section 2 and is carried out using delicate induction arguments.

THEOREM 2.1. G_n can be factorized as $G_n = L_n U_n$, where $L_n = [L_n(i,j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]$ is an upper triangular matrix, whose entries are defined as follows:

$$L_n(i,j) = \begin{cases} 1, & i = j; \\ 0, & i < j; \\ (\frac{x_i}{x_1})^{a_1}, & j = 1, i \geq 2; \\ (\frac{x_i}{x_j})^{a_1} \frac{S_{\{x_j \to x_i\}}(A_j)}{A_j}, & i \geq j+1, j \geq 2. \end{cases};$$

and

$$U_n(i,j) = \begin{cases} x_1^{a_j}, & i = 1; \\ 0, & i > j; \\ x_i^{a_1} B_i, & i = j \geq 2; \\ x_i^{a_1} S^{\{a_i \to a_j\}}(B_i), & j \geq i+1, i \geq 2. \end{cases};$$

where A_i, B_j, and the notations $S_{\{x_{k-1} \to x_k\}}$, $S^{\{a_{k-1} \to a_k\}}$ are defined recursively for $i, j \geq 2$:

$$A_2 = B_2 = x_2^{a_2 - a_1} - x_1^{a_2 - a_1};$$

$$A_k = \{S^{\{a_{k-1} \to a_k\}}_{\{x_{k-1} \to x_k\}}(A_{k-1})\} A_{k-1} - S_{\{x_{k-1} \to x_k\}}(A_{k-1}) S^{\{a_{k-1} \to a_k\}}(A_{k-1}), k \geq 3;$$

$$B_k = S^{\{a_{k-1}\to a_k\}}_{\{x_{k-1}\to x_k\}}(B_{k-1}) - \ \{S^{\{a_{k-1}\to a_k\}}(B_{k-1})\}\left(\frac{S_{\{x_{k-1}\to x_k\}}(A_{k-1})}{A_{k-1}}\right), k \geq 3;$$

$$S_{\{x_{k-1}\to x_k\}}(A_{k-1}) \ := \ x_k \text{ substitutes } x_{k-1} \text{ in } A_{k-1};$$

$$S^{\{a_{k-1}\to a_k\}}(A_{k-1}) \ := \ a_k \text{ substitutes } a_{k-1} \text{ in } A_{k-1}.$$

PROOF. We use mathematical induction on n, the size of G_n. The case where $n = 2$ follows from

$$L_2U_2 = \begin{bmatrix} 1 & 0 \\ (\frac{x_2}{x_1})^{a_1} & 1 \end{bmatrix} \begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ 0 & x_2^{a_1}(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \end{bmatrix} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ x_2^{a_1} & x_2^{a_2} \end{bmatrix} = G_2.$$

Assume $G_k = L_kU_k$ holds, we want to prove $G_{k+1} = L_{k+1}L_{k+1}$. By induction hypothesis, we know that

$$G_k(k,l) = \sum_{m=1}^{l} L_k(k,m)U_k(m,l), 1 \leq l \leq k, \tag{1}$$

$$G_k(l,k) = \sum_{m=1}^{l} L_k(l,m)U_k(m,k), 1 \leq l \leq k. \tag{2}$$

It is sufficient to show that

$$G_{k+1}(k+1,l) = \sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,l), 1 \leq l \leq k,$$

and

$$G_{k+1}(l,k+1) = \sum_{m=1}^{k+1} L_{k+1}(l,m)U_{k+1}(m,k+1), 1 \leq l \leq k+1.$$

Firstly, we find that

$$\begin{aligned} \sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,1) &= L_{k+1}(k+1,1)U_{k+1}(1,1) \\ &= \left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_1} \\ &= x_{k+1}^{a_1} \\ &= G_{k+1}(k+1,1), \end{aligned}$$

and by (1), we get

$$\begin{aligned}\sum_{m=1}^{k} L_k(k,m)U_k(m,l) &= \sum_{m=1}^{l} L_k(k,m)U_k(m,l) \\ &= \left(\frac{x_k}{x_1}\right)^{a_1} \times x_1^{a_l} \\ &\quad + \sum_{m=2}^{l} \left(\frac{x_k}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_l\}}(B_m) \\ &= G_k(k,l) \\ &= x_k^{a_l},\end{aligned}$$

for $2 \le l \le k-1$, so

$$\begin{aligned}&\sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,l) \\ =\ &\sum_{m=1}^{l} L_{k+1}(k+1,m)U_{k+1}(m,l) \\ =\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_l} + \sum_{m=2}^{l} \left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_l\}}(B_m) \\ =\ &x_{k+1}^{a_l} \\ =\ &G_{k+1}(k+1,l),\end{aligned}$$

for $2 \le l \le k-1$. Next, we have

$$\begin{aligned}\sum_{m=1}^{k+1} L_{k+1}(1,m)U_{k+1}(m,k+1) &= L_{k+1}(1,1)U_{k+1}(1,k+1) \\ &= 1 \times x_1^{a_{k+1}} \\ &= x_1^{a_{k+1}} \\ &= G_{k+1}(1,k+1),\end{aligned}$$

and by (2), we can get

$$\begin{aligned}&\sum_{m=1}^{k} L_k(l,m)U_k(m,k) \\ =\ &\sum_{m=1}^{l} L_k(l,m)U_k(m,k) \\ =\ &(\frac{x_l}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{l} (\frac{x_l}{x_m})^{a_1} \frac{S_{\{x_m \to x_l\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_k\}}(B_m) \\ =\ &G_k(l,k) \\ =\ &x_l^{a_k},\end{aligned}$$

for $2 \leq l \leq k-1$, so

$$\begin{aligned}
&\sum_{m=1}^{k+1} L_{k+1}(l,m)U_{k+1}(m,k+1)\\
=\;& \sum_{m=1}^{l} L_{k+1}(l,m)U_{k+1}(m,k+1)\\
=\;& \left(\frac{x_l}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{l} \left(\frac{x_l}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_l\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_{k+1}\}}(B_m)\\
=\;& x_l^{a_{k+1}} = G_{k+1}(l,k+1),
\end{aligned}$$

for $2 \leq l \leq k-1$; let $l = k$ in (2), we establish the following equation:

$$\begin{aligned}
x_k^{a_k} \;=\;& \sum_{m=1}^{k} L_k(k,m)U_k(m,k)\\
=\;& \left(\frac{x_k}{x_1}\right)^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1} \left(\frac{x_k}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_k\}}(B_m)\\
&+1 \times x_k^{a_1} B_k. \qquad (3)
\end{aligned}$$

Altogether, we conclude that

$$\begin{aligned}
&\sum_{m=1}^{k+1} L_{k+1}(k,m)U_{k+1}(m,k+1)\\
=\;& \sum_{m=1}^{k} L_{k+1}(k,m)U_{k+1}(m,k+1)\\
=\;& \left(\frac{x_k}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1} \left(\frac{x_k}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_{k+1}\}}(B_m)\\
&+1 \times x_k^{a_1} \times S^{\{a_k \to a_{k+1}\}} B_k\\
=\;& x_k^{a_{k+1}}\\
=\;& G_{k+1}(k,k+1).
\end{aligned}$$

On the other hand, we will prove that

$$\begin{aligned}
x_{k+1}^{a_k} \;=\;& \left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1} \left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m \to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_k\}}(B_m)\\
&+x_{k+1}^{a_1} \left(\frac{S_{\{x_k \to x_{k+1}\}}(A_k)}{A_k}\right) \times B_k.
\end{aligned}$$

In fact, application of $S_{\{x_k \to x_{k+1}\}}$ on both sides of (3) results in the latter equation except the last term being $1 \times S_{\{x_k \to x_{k+1}\}}(x_k^{a_1} B_k)$. Thus the validity of the latter equation is based on the fact that

$$A_k \times S_{\{x_k \to x_{k+1}\}}(B_k) = B_k \times S_{\{x_k \to x_{k+1}\}}(A_k)$$

which can be proved by induction on k as follows. The case $k = 2$ is trivial, since $A_2 = B_2$. Assume the induction hypothesis

$$A_k \times S_{\{x_k \to x_{k+1}\}}(B_k) = B_k \times S_{\{x_k \to x_{k+1}\}}(A_k)$$

holds. Then by applying $S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}} S_{\{x_{k+1} \to x_{k+2}\}}$, $S^{\{a_k \to a_{k+1}\}}$ and $S^{\{a_k \to a_{k+1}\}}_{\{x_{k+1} \to x_{k+2}\}}$ to both sides of the above equation respectively, we obtain

$$S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(B_k) = S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(A_k); \tag{4}$$

$$S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(B_k) = S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) \times S^{\{a_k \to a_{k+1}\}}(A_k); \tag{5}$$

$$S^{\{a_k \to a_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(B_k) = S^{\{a_k \to a_{k+1}\}}(B_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(A_k). \tag{6}$$

Now we prove the induction step

$$A_{k+1} \times S_{\{x_{k+1} \to x_{k+2}\}}(B_{k+1}) = B_{k+1} \times S_{\{x_{k+1} \to x_{k+2}\}}(A_{k+1}),$$

which, by using the definitions of A_{k+1} and B_{k+1}, becomes

$$\begin{aligned}
& (\{S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k)\}A_k - S_{\{x_k \to x_{k+1}\}}(A_k) S^{\{a_k \to a_{k+1}\}}(A_k)) \\
& \times S_{\{x_{k+1} \to x_{k+2}\}} \left(S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) - \{S^{\{a_k \to a_{k+1}\}}(B_k)\} \left(\frac{S_{\{x_k \to x_{k+1}\}}(A_k)}{A_k} \right) \right) \\
= \; & \left(S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) - \{S^{\{a_k \to a_{k+1}\}}(B_k)\} \left(\frac{S_{\{x_k \to x_{k+1}\}}(A_k)}{A_k} \right) \right) \\
& \times S_{\{x_{k+1} \to x_{k+2}\}} (\{S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k)\}A_k - S_{\{x_k \to x_{k+1}\}}(A_k) S^{\{a_k \to a_{k+1}\}}(A_k)),
\end{aligned}$$

and further calculations yield the following four-term equation

$$\begin{aligned}
& A_k^2 \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(B_k) \\
& - A_k \times S^{\{a_k \to a_{k+1}\}}(B_k) \times S_{\{x_k \to x_{k+2}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(A_k) \\
& - A_k \times S_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(B_k) \\
& + S_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(A_k) \times S_{\{x_k \to x_{k+2}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(B_k) \\
= \; & A_k^2 \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(A_k) \\
& - A_k \times S_{\{x_k \to x_{k+2}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k) \\
& - A_k \times S^{\{a_k \to a_{k+1}\}}(B_k) \times S_{\{x_k \to x_{k+1}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+2}\}}(A_k) \\
& + S^{\{a_k \to a_{k+1}\}}(B_k) \times S_{\{x_k \to x_{k+1}\}}(A_k) \times S_{\{x_k \to x_{k+2}\}}(A_k) \times S^{\{a_k \to a_{k+1}\}}(A_k).
\end{aligned} \tag{7}$$

Now the first terms on both sides of (7) are equal due to (4), the second terms to (5) and the third terms to (6), while the fourth terms are exactly equal. Thus we have completed the proof.

Thus, we have

$$\begin{aligned}
&\sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,k)\\
=\ &\sum_{m=1}^{k} L_{k+1}(k+1,m)U_{k+1}(m,k)\\
=\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1}\left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m\to a_k\}}(B_m)\\
&+\left(\frac{x_{k+1}}{x_k}\right)^{a_1}\left(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}\right)\times x_k^{a_1}B_k\\
=\ &x_{k+1}^{a_k}\\
=\ &G_{k+1}(k+1,k),
\end{aligned}$$

and by (3),

$$\begin{aligned}
&\sum_{m=1}^{k+1} L_{k+1}(k,m)U_{k+1}(m,k+1)\\
=\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k}\left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m\to a_{k+1}\}}(B_m)\\
&+1\times x_{k+1}^{a_1}B_{k+1}\\
=\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1}\left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m\to a_{k+1}\}}(B_m)\\
&+\left(\frac{x_{k+1}}{x_k}\right)^{a_1}\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}\times x_k^{a_1}S^{\{a_k\to a_{k+1}\}}(B_k) + x_{k+1}^{a_1}B_{k+1}\\
=\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1}\left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m\to a_{k+1}\}}(B_m)\\
&+x_{k+1}^{a_1}\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}\times S^{\{a_k\to a_{k+1}\}}(B_k)\\
&+x_{k+1}^{a_1}\times\left(S^{\{a_k\to a_{k+1}\}}_{\{x_k\to x_{k+1}\}}(B_k) - S^{\{a_k\to a_{k+1}\}}(B_k)\left(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{(A_k)}\right)\right)\\
=\ &\left(\frac{x_{k+1}}{x_1}\right)^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1}\left(\frac{x_{k+1}}{x_m}\right)^{a_1} \frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m\to a_{k+1}\}}(B_m)\\
&+x_{k+1}^{a_1}\times S^{\{a_k\to a_{k+1}\}}_{\{x_k\to x_{k+1}\}}(B_k)\\
=\ &x_{k+1}^{a_{k+1}} = G_{k+1}(k+1,k+1).
\end{aligned}$$

Thus we have completed the proof.

To illustrate our result, we first give an example of the explicit factorization of G_4, and then using Mathematica we provide a program to verify the correctness of our result in a special case.

EXAMPLE 2.2. Let $n = 4$. Then $G_4 = L_4 U_4$, where

$$G_4 = \begin{bmatrix} {x_1}^{a_1} & {x_1}^{a_2} & {x_1}^{a_3} & {x_1}^{a_4} \\ {x_2}^{a_1} & {x_2}^{a_2} & {x_2}^{a_3} & {x_2}^{a_4} \\ {x_3}^{a_1} & {x_3}^{a_2} & {x_3}^{a_3} & {x_3}^{a_4} \\ {x_4}^{a_1} & {x_4}^{a_2} & {x_4}^{a_3} & {x_4}^{a_4} \end{bmatrix}, L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (\frac{x_2}{x_1})^{a_1} & 1 & 0 & 0 \\ (\frac{x_3}{x_1})^{a_1} & (\frac{x_3}{x_2})^{a_1}(\frac{D^{a_2,a_1}_{x_3,x_1}}{D^{a_2,a_1}_{x_2,x_1}}) & 1 & 0 \\ (\frac{x_4}{x_1})^{a_1} & (\frac{x_4}{x_2})^{a_1}(\frac{D^{a_2,a_1}_{x_4,x_1}}{D^{a_2,a_1}_{x_2,x_1}}) & L_4(4,3) & 1 \end{bmatrix},$$

$$U_4 = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ 0 & x_2^{a_1}(D^{a_2,a_1}_{x_2,x_1}) & x_2^{a_1}(D^{a_3,a_1}_{x_2,x_1}) & x_2^{a_1}(D^{a_4,a_1}_{x_2,x_1}) \\ 0 & 0 & x_3^{a_1}((D^{a_3,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})}) & x_3^{a_1}((D^{a_4,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})}) \\ 0 & 0 & 0 & U_4(4,4) \end{bmatrix},$$

and

$$L_4(4,3) = \left(\frac{x_4}{x_3}\right)^{a_1} \left[\frac{(D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})}\right],$$

$$\begin{aligned} U_4(4,4) \quad = \quad & \Bigg\{x_4^{a_1}\Bigg((D^{a_4,a_1}_{x_4,x_1}) - \frac{(D^{a_2,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} - \Bigg((D^{a_4,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})}\Bigg) \\ & \times \left[\frac{(D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})}\right]\Bigg)\Bigg\}. \end{aligned}$$

Here and in the following Examples 2.4 and 2.6, to simplify our notation, we set $D^{a_i,a_j}_{x_m,x_n} := x_m^{a_i-a_j} - x_n^{a_i-a_j}$.

An explicit calculation using Mathematica. Let $n = 4$, $x_1 = x, x_2 = y, x_3 = z, x_4 = w$ and $a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 6$, then the Mathematica program on $L_4 \times U_4$

$FullSimplify[\{\{1,0,0,0\},\{1,1,0,0\},\{1,(z^2-x^2)/(y^2-x^2),1,0\},\{1,(w^2-x^2)/(y^2-x^2),((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4)),1\}\}.\{\{1,x^2,x^4,x^6\},\{0,(y^2-x^2),(y^4-x^4),(y^6-x^6)\},\{0,0,((z^4-x^4)-((z^2-x^2)(y^4-x^4))/(y^2-x^2)),((z^6-x^6)-((z^2-x^2)(y^6-x^6))/(y^2-x^2))\},\{0,0,0,((w^6-x^6)-((w^2-x^2)(y^6-x^6))/(y^2-x^2))-((z^6-x^6)-((z^2-x^2)(y^6-x^6))/(y^2-x^2))((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4))\}\}]$

produces the result

$$\begin{bmatrix} 1 & x^2 & x^4 & x^6 \\ 1 & y^2 & y^4 & y^6 \\ 1 & z^2 & z^4 & z^6 \\ 1 & w^2 & w^4 & w^6 \end{bmatrix}.$$

Since the determinant of a triangular matrix is the product of the entries of the main diagonal, as a first by-product, we get the following immediate corollary which provides a recursive formula for the determinant of G_n.

COROLLARY 2.3. The determinant of G_n is as follows:

$$\det G_n = \prod_{1\le i\le n} U_n(i,i) = x_1^{a_1} \times \prod_{2\le i\le n} x_i^{a_1} B_i.$$

EXAMPLE 2.4. Let $n = 4$. Then

$$\begin{aligned}
\det G_4 &= \prod_{1\le i\le 4} U_n(i,i) = x_1^{a_1} \times \prod_{2\le i\le 4} x_i^{a_1} B_i \\
&= x_1^{a_1} x_2^{a_1} (D^{a_2,a_1}_{x_2,x_1}) x_3^{a_1} \left((D^{a_3,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} \right) \\
&\quad \times \left\{ x_4^{a_1} \left((D^{a_4,a_1}_{x_4,x_1}) - \frac{(D^{a_2,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} - \left((D^{a_4,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} \right) \right. \right. \\
&\quad \times \left. \left. \left[\frac{(D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})} \right] \right) \right\}.
\end{aligned}$$

For any $n \times n$ matrix A, let $\tilde{A}_{ij}$ be the matrix obtained from A by deleting the i-th row and the j-th column. Then

$$A^{-1} = Transpose\ of\ \frac{(-1)^{i+j} det(\tilde{A}_{ij})}{det(A)}.$$

Because $(\tilde{G}_n)_{ji}$ is still a totally positive generalized Vandermonde matrix, $det(\tilde{G}_n)_{ji}$ can be computed by Corollary 2.3. Basing on the fact and the above formula, we establish the second by-product:

COROLLARY 2.5. The entry of the inverse of G_n is $G_n^{-1}(i,j) = \frac{(-1)^{j+i} \det((\tilde{G}_n)_{ji})}{\det(G_n)}$.

EXAMPLE 2.6. Let $n = 4$. Then G_4^{-1} is given by

$$\begin{aligned}
&\frac{1}{x_1^{a_1} x_2^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1}))} \\
&\times \frac{1}{(D^{a_4,a_1}_{x_4,x_1}) - \frac{(D^{a_2,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} - \left((D^{a_4,a_1}_{x_3,x_1}) - \frac{(D^{a_2,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})}{(D^{a_2,a_1}_{x_2,x_1})} \right) \times \left[\frac{(D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1})}{(D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1})} \right]} \\
&\times \begin{bmatrix}
x_2^{a_2} x_3^{a_2} x_4^{a_2} ((D^{a_4,a_2}_{x_4,x_2})(D^{a_3,a_2}_{x_3,x_2}) - (D^{a_3,a_2}_{x_4,x_2})(D^{a_4,a_2}_{x_3,x_2})) & -x_1^{a_2} x_3^{a_2} x_4^{a_2} ((D^{a_4,a_2}_{x_4,x_1})(D^{a_3,a_2}_{x_3,x_1}) - (D^{a_3,a_2}_{x_4,x_1})(D^{a_4,a_2}_{x_3,x_1})) & x_1^{a_2} x_2^{a_2} x_4^{a_2} ((D^{a_4,a_2}_{x_4,x_1})(D^{a_3,a_2}_{x_2,x_1}) - (D^{a_3,a_2}_{x_4,x_1})(D^{a_4,a_2}_{x_2,x_1})) & -x_1^{a_2} x_2^{a_2} x_3^{a_2} ((D^{a_4,a_2}_{x_3,x_1})(D^{a_3,a_2}_{x_2,x_1}) - (D^{a_3,a_2}_{x_3,x_1})(D^{a_4,a_2}_{x_2,x_1})) \\
-x_2^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_2})(D^{a_3,a_1}_{x_3,x_2}) - (D^{a_3,a_1}_{x_4,x_2})(D^{a_4,a_1}_{x_3,x_2})) & x_1^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_3,x_1}) - (D^{a_3,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_3,x_1})) & -x_1^{a_1} x_2^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1}) - (D^{a_3,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_2,x_1})) & x_1^{a_1} x_2^{a_1} x_3^{a_1} ((D^{a_4,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1}) - (D^{a_3,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})) \\
x_2^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_2})(D^{a_2,a_1}_{x_3,x_2}) - (D^{a_2,a_1}_{x_4,x_2})(D^{a_4,a_1}_{x_3,x_2})) & -x_1^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_3,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_3,x_1})) & x_1^{a_1} x_2^{a_1} x_4^{a_1} ((D^{a_4,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_4,a_1}_{x_2,x_1})) & -x_1^{a_1} x_2^{a_1} x_3^{a_1} ((D^{a_4,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_4,a_1}_{x_2,x_1})) \\
-x_2^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_3,a_1}_{x_4,x_2})(D^{a_2,a_1}_{x_3,x_2}) - (D^{a_2,a_1}_{x_4,x_2})(D^{a_3,a_1}_{x_3,x_2})) & x_1^{a_1} x_3^{a_1} x_4^{a_1} ((D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_3,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_3,x_1})) & -x_1^{a_1} x_2^{a_1} x_4^{a_1} ((D^{a_3,a_1}_{x_4,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_4,x_1})(D^{a_3,a_1}_{x_2,x_1})) & x_1^{a_1} x_2^{a_1} x_3^{a_1} ((D^{a_3,a_1}_{x_3,x_1})(D^{a_2,a_1}_{x_2,x_1}) - (D^{a_2,a_1}_{x_3,x_1})(D^{a_3,a_1}_{x_2,x_1}))
\end{bmatrix}.
\end{aligned}$$

As a last by-product, due to Corollary 2.3, we give the following recursive formula of the Schur function $s_\lambda(x_1, x_2, \cdots, x_n)$ and we illustrate the result by an example.

COROLLARY 2.7. The Schur function $s_\lambda(x_1, x_2, \cdots, x_n)$ can be expressed as

$$s_\lambda(x_1, x_2, \cdots, x_n) = \frac{x_1^{a_1} \times \prod_{2\le i\le n} x_i^{a_1} B_i}{\prod_{1\le i<j\le n}(x_j - x_i)}.$$

EXAMPLE 2.8. Let $\lambda = (7,5,3,1)$ and $n = 4$. Then $(a_1, a_2, a_3, a_4) = (1,4,7,10)$ and

$$\begin{aligned}
\det G_{\{4;1,4,7,10\}} &= x_1^{a_1} \times \prod_{2\le i\le 4} x_i^{a_1} B_i \\
&= x_1 x_2 (x_2^3 - x_1^3) x_3 \left((x_3^6 - x_1^6) - \frac{(x_3^3 - x_1^3)(x_2^6 - x_1^6)}{(x_2^3 - x_1^3)} \right) \\
&\quad \times x_4 \left((x_4^9 - x_1^9) - \frac{(x_4^3 - x_1^3)(x_2^9 - x_1^9)}{(x_2^3 - x_1^3)} \right. \\
&\quad - \left((x_3^9 - x_1^9) - \frac{(x_3^3 - x_1^3)(x_2^9 - x_1^9)}{(x_2^3 - x_1^3)} \right) \\
&\quad \times \left. \left(\frac{(x_4^6 - x_1^6)(x_2^3 - x_1^3) - (x_4^3 - x_1^3)(x_2^6 - x_1^6)}{(x_3^6 - x_1^6)(x_2^3 - x_1^3) - (x_3^3 - x_1^3)(x_2^6 - x_1^6)} \right) \right) \\
&= x_1 x_2 x_3 x_4 (x_4^2 + x_4 x_3 + x_3^2)(x_4^2 + x_4 x_2 + x_2^2)(x_4^2 + x_4 x_1 + x_1^2) \\
&\quad \times (x_3^2 + x_3 x_2 + x_2^2)(x_3^2 + x_3 x_1 + x_1^2)(x_2^2 + x_2 x_1 + x_1^2) \\
&\quad \times \prod_{1\le i<j\le 4} (x_j - x_i),
\end{aligned}$$

so by Corollary 2.7, we see that $s_{(7,5,3,1)}(x_1, x_2, x_3, x_4)$ is equal to

$$\begin{aligned}
&x_1 x_2 x_3 x_4 (x_4^2 + x_4 x_3 + x_3^2)(x_4^2 + x_4 x_2 + x_2^2) \\
&\times (x_4^2 + x_4 x_1 + x_1^2)(x_3^2 + x_3 x_2 + x_2^2)(x_3^2 + x_3 x_1 + x_1^2)(x_2^2 + x_2 x_1 + x_1^2).
\end{aligned}$$

As we can see, there are $3^6 = 729$ semistandard $(7,5,3,1)$ tableaux, it seems not easy to write out all of the semistandard $(7,5,3,1)$ tableaux.

References

[1] J. Demmel and P. Koev, The accurate and efficient solution of a totally positive generalized Vandermonde linear system, SIAM J. Matrix Anal. Appl., 27(1)(2005), 142–152.

[2] H.-C. Li and E.-T. Tan, On a special generalized Vandermonde matrix and its LU factorization, to appear in Taiwanese Journal of Mathematics.