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On The Propagation Of Plane Waves At A Pure Shear Interface*

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Abstract

Propagation of elastic waves at a pure shear interface of two half-spaces with different strain-energy functions is studied. An interesting result of the existence of two interfacial waves (with no transmitted waves) in the upper half-space is found. The wave amplitudes are illustrated graphically highlighting the dependence on the finite pure shear deformation and the angle of incidence.

1 Introduction

In [1] Hussain and Ogden considered the effect of simple shear deformation on the reflection and transmission of homogeneous plane waves at the boundary between two half-spaces consist of the same material and corresponding to the same strain-energy functions (see also the related paper [2]). In the present paper the effect of *pure shear* on the reflection and transmission of plane waves at the boundary between two half-spaces corresponding to *different strain-energy* functions to consist of same incompressible isotropic elastic material is considered.

The required equations and notations are summarized in Section 2 and in the Subsections the propagation of plane harmonic waves is discussed with reference to the *slowness curves* appropriate for the two distinct classes of strain-energy functions.

For the *mixed case* of the strain-energy functions, the method for finding the amplitudes of the reflected, transmitted and interfacial waves is discussed in Section 3.1. For each angle of incidence a single reflected wave, with angle of reflection equal to the angle of incidence, is generated when a homogeneous plane (SV) wave is incident on the boundary from one half-space, and it is accompanied by an interfacial wave. It is shown that under a certain restriction on the state of deformation a transmitted (homogeneous plane SV) wave and an interfacial wave are generated for all angles of incidence. When this restriction does not hold there exists a critical angle such that (a) in $x_2 < 0$, above which, there is a reflected wave and an interfacial wave but below which there are two reflected waves (and no interfacial waves), *and* (b) in $x_2 > 0$,

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above which, there is again a transmitted and an interfacial wave but below which there are two interfacial waves (and no transmitted wave).

The theory in Section 3 is illustrated in Section 4 using graphical results to show the dependence of the amplitudes of the waves on the angle of incidence for representative values of the deformation parameters.

2 Basic Equations

Consider an incompressible isotropic elastic material subject to pure shear. Let λ_1, λ_2, λ_3 denote the principal stretches of the deformation. Then, the incompressibility condition is expressed as

$$\lambda_1\lambda_2\lambda_3 = 1.$$

If $\sigma_i\,(i = 1, 2, 3)$ are the principal Cauchy stresses then the pure shear deformation confined to the (1-2)-plane is given by

$$\lambda_1 = \lambda \neq 1,\ \ \lambda_2 = \lambda^{-1},\ \lambda_3 = 1 \text{ with } \sigma_1 \neq 0,\ \sigma_2 = 0,$$

where a non-vanishing stress σ_3 is required to maintain $\lambda_3 = 1$.

The equation of motion in terms of a scalar function, $\psi(x_1, x_2, t)$ say, is given by

$$\alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}), \tag{1}$$

as in [1], where $,i$ denotes $\partial/\partial x_i$, $i \in \{1, 2\}$, ρ is the mass density, a superposed dot indicates the material time derivative and the constants α, β, γ are defined by

$$\alpha = \mathcal{A}_{01212}, \quad \gamma = \mathcal{A}_{02121}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}. \tag{2}$$

In (2) $\mathcal{A}_{0jilk}$ are the components of the fourth-order tensor $\boldsymbol{\mathcal{A}_0}$ of instantaneous elastic moduli (see, for example, Ogden [3]).

The shear and normal components of the incremental nominal traction Σ_{21}, Σ_{22} on a plane $x_2 =$ constant are expressible in terms of ψ through

$$\begin{aligned} \Sigma_{21} &= \gamma\psi_{,22} - (\gamma - \sigma_2)\psi_{,11}, \\ -\Sigma_{22,1} &= (2\beta + \gamma - \sigma_2)\psi_{,112} + \gamma\psi_{,222} - \rho\ddot{\psi}_{,2}. \end{aligned} \tag{3}$$

2.1 Plane Waves

Consider time-harmonic homogeneous plane waves of the form

$$\psi = A\exp[ik(x_1\cos\theta + x_2\sin\theta - ct)], \tag{4}$$

where A is a constant, c (> 0) the wave speed, k (> 0) the wave number and ($\cos\theta$, $\sin\theta$) the direction cosines of the direction of propagation of the wave in the (x_1, x_2)-plane. Substitution of (4) into (1) gives

$$\alpha\cos^4\theta + 2\beta\sin^2\theta\cos^2\theta + \gamma\sin^4\theta = \rho c^2. \tag{5}$$

Equation (5) is a relationship between the wave speed and the propagation direction in the (x_1, x_2)-plane and is called the propagation condition. The material constants are taken to satisfy the strong ellipticity inequalities

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}, \tag{6}$$

and it is clear from (5) that $\rho c^2 > 0$ if and only if (6) hold.

Similarly, from (1), for an inhomogeneous plane wave of the form

$$\psi = \hat{A}\exp[ik'(x_1 - imx_2 - c't)], \tag{7}$$

we obtain

$$\alpha - 2\beta m^2 + \gamma m^4 = \rho(1 - m^2){c'}^2, \tag{8}$$

which relates the wave speed c' to the 'inhomogeneity factor' m. Note that the wave decays exponentially as $x_2 \to -\infty(+\infty)$ provided m has positive (negative) real part.

Consider two distinct cases corresponding to different strain-energy functions. For these either $2\beta = \alpha + \gamma$ or $2\beta \neq \alpha + \gamma$.

2.1.1 Case A: $2\beta = \alpha + \gamma$

For this case equations (5) and (8) reduce to

$$\alpha\cos^2\theta + \gamma\sin^2\theta = \rho c^2 \tag{9}$$

and

$$(m^2 - 1)(\alpha - \gamma m^2 - \rho{c'}^2) = 0 \tag{10}$$

respectively.

In terms of the *slowness vector* (s_1, s_2) defined by

$$(s_1, s_2) = (\cos\theta, \sin\theta)/c$$

equation (9) becomes the *slowness* curve (by using $\alpha/\gamma = \lambda^4$)

$$\lambda^4{s_1}^2 + {s_2}^2 = \overline{\rho}, \tag{11}$$

in the (s_1, s_2)-space, where $\overline{\rho}$ is defined by

$$\overline{\rho} = \rho/\gamma. \tag{12}$$

By using the dimensionless notation $(\overline{s}_1, \overline{s}_2)$ defined by

$$(\overline{s}_1, \overline{s}_2) \equiv (s_1, s_2)/\sqrt{\overline{\rho}}, \tag{13}$$

we can write (11) as

$$\lambda^4{\overline{s}_1}^2 + {\overline{s}_2}^2 = 1. \tag{14}$$

2.1.2 Case B: $2\beta \neq \alpha + \gamma$

In this case we take the strain-energy function to satisfy $\beta = \sqrt{\alpha\gamma}$ which was used by Hussain and Ogden in [1]. Then (5) takes the form

$$[\sqrt{\alpha}\cos^2\theta + \sqrt{\gamma}\sin^2\theta]^2 = \rho c^2 \tag{15}$$

and (8) becomes

$$(\sqrt{\alpha} - \sqrt{\gamma}m^2)^2 = \rho(1 - m^2)c'^2. \tag{16}$$

The slowness curve corresponding to (15) is given by

$$[\lambda^2\overline{s}_1^2 + \overline{s}_2^2]^2 = \overline{s}_1{}^2 + \overline{s}_2{}^2, \tag{17}$$

in dimensionless form with the notation (13) and $\overline{\rho}$ defined by (12). We now show graphically the dependence of the slowness curves on λ for both classes of strain-energy functions in $(\overline{s}_1, \overline{s}_2)$-space with reference to (14) and (17) (See Figs. 1-2).

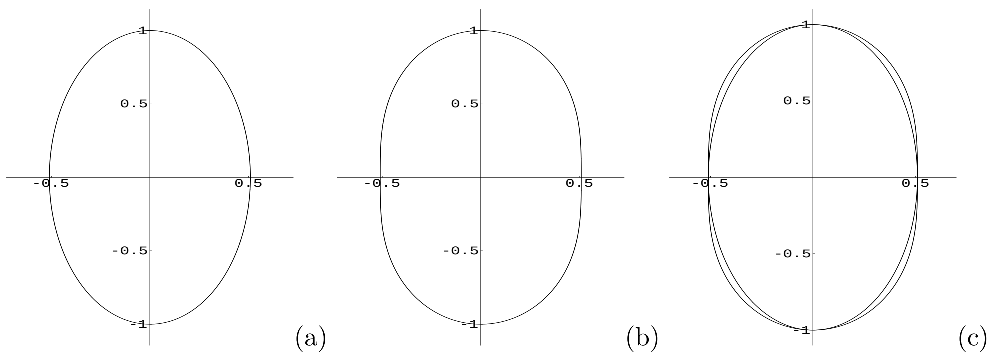

Figure 1: Slowness curves in $(\overline{s}_1, \overline{s}_2)$-space for $\lambda = 1.4$ with (a) $2\beta = \alpha + \gamma$, (b) $2\beta \neq \alpha + \gamma$, (c) the superposition of Figs. in (a) and (b).

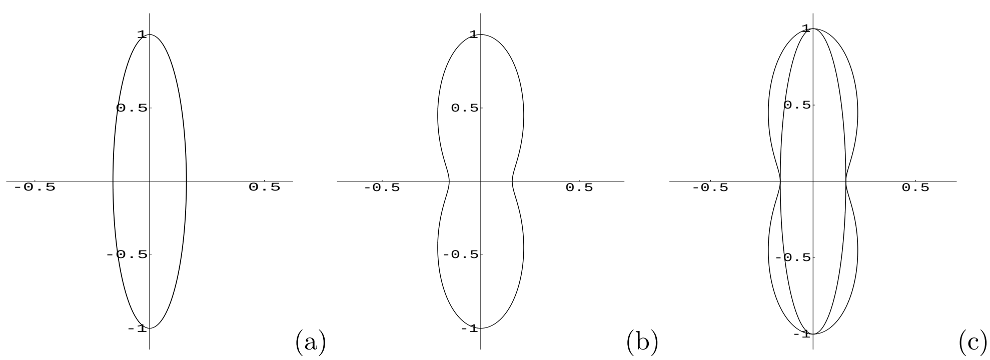

Figure 2: Slowness curves in $(\overline{s}_1, \overline{s}_2)$-space for $\lambda = 2.5$ with (a) $2\beta = \alpha + \gamma$, (b) $2\beta \neq \alpha + \gamma$, (c)the superposition of Figs. in (a) and (b).

3 Reflection and transmission at the interface

Consider the case in which a half-space defined by $x_2 < 0$ is joined to a half-space $x_2 > 0$. Both are subjected to pure shear deformation. The boundary conditions corresponding to continuous incremental displacement across the interface $x_2 = 0$ in terms of the scalar functions ψ and ψ^*, take the forms

$$\psi_{,1} = \psi^*_{,1},\ \psi_{,2} = \psi^*_{,2}, \tag{18}$$

respectively on $x_2 = 0$, where an asterisk signifies a quantity in $x_2 > 0$.

In terms of ψ, boundary conditions corresponding to continuous incremental traction, across the interface $x_2 = 0$ are recast through

$$\Sigma_{21} = \Sigma^*_{21}, \quad \Sigma_{22,1} = \Sigma^*_{22,1},$$

on use of (3), in the forms

$$\begin{aligned} \psi_{,11} - \psi_{,22} = \psi^*_{,11} - \psi^*_{,22}, \\ (2\beta + \gamma)(\psi_{,112} - \psi^*_{,112}) + \gamma(\psi_{,222} - \psi^*_{,222}) - \rho(\ddot{\psi}_{,2} - \ddot{\psi}^*_{,2}) = 0. \end{aligned} \tag{19}$$

Because of the symmetry of slowness curves, with respect to the normal direction to the interface, the solution comprising the incident wave, a reflected wave and an interfacial wave in $x_2 < 0$ is written as

$$\begin{aligned} \psi = A\exp[ik(x_1\cos\theta + x_2\sin\theta - ct)] + AR\exp[ik(x_1\cos\theta - x_2\sin\theta - ct)] \\ + AR'\exp[ik'(x_1 - imx_2 - c't)], \end{aligned} \tag{20}$$

where R is the reflection coefficient and R' measures the amplitude of the interfacial wave. The notations k', m, c' are as used in (7) and m has positive real part.

Accordingly, in $x_2 > 0$, the wave solution may be written

$$\psi^* = AR^*\exp[ik^*(x_1\cos\theta^* + x_2\sin\theta^* - c^*t)] + AR^{*'}\exp[ik^{*'}(x_1 + im^*x_2 - c^{*'}t)], \tag{21}$$

comprising a transmitted and an interfacial wave, where R^* is the transmission coefficient and $R^{*'}$ is the analogue of R' for $x_2 > 0$. Note that the interfacial wave decays as $x_2 \to \infty$ provided m^* has positive real part.

Here Snell's law takes the form

$$\cos\theta/c = 1/c^{'} = \cos\theta^*/c^* = 1/c^{*'}. \tag{22}$$

(22) states in particular, that the first components of the slowness vectors for each homogeneous plane wave interacting at the boundary $x_2 = 0$ are equal.

Thus, by reference to the slowness curves (superimposed) as exemplified in Fig. 1(c) and Fig. 2(c), the range of angles of incidence for which a transmitted wave exists can be identified. In Fig. 1(c), for example, if the *outer curve* corresponds to $x_2 < 0$ there is, for every angle of incidence (i.e. for every s_1 associated with the curve) a point on the inner curve (corresponding to $x_2 > 0$), and hence a transmitted wave.

In Fig. 2(c) on the other hand, there are values of s_1 on the slowness (outer) curve for $x_2 < 0$ for which there are no corresponding values on the inner slowness curve (corresponding to $x_2 > 0$), and therefore a range of angles of incidence for which no transmitted wave exists. This will be discussed further in Section 3.1.

We now examine here the case in which $2\beta \neq \alpha + \gamma\,(x_2 < 0)$, $2\beta = \alpha + \gamma\,(x_2 > 0)$.

3.1 $2\beta \neq \alpha + \gamma\,(x_2 < 0)$, $2\beta = \alpha + \gamma\,(x_2 > 0)$

In this case we see from (16), after using Snell's law $\cos\theta/c = 1/c'$, we have

$$(m^2 + t^2)[m^2(1 + t^2) - t^2 + \lambda^2(\lambda^2 - 2)] = 0, \tag{23}$$

where $t = \tan\theta$. Note that t should be distinguished from the time variable t used earlier. $m = +it$ and $m = -it$ are the solutions of (23) corresponding to the incident and reflected waves respectively. The other solutions are

$$m = \pm\sqrt{1 - (\lambda^2 - 1)^2/(1 + t^2)}. \tag{24}$$

If $\lambda \leq \sqrt{2}$ the m is real for all θ and the positive solution of (24) corresponds to an interfacial wave in $x_2 < 0$. If $\lambda > \sqrt{2}$ then there is a critical value of θ, θ_c say, for which $m = 0$ and this is given by

$${t_c}^2 = \lambda^2(\lambda^2 - 2), \tag{25}$$

where the notation $t_c = \tan\theta_c$ is used. It follows that m is real for $\theta_c \leq \theta \leq \pi/2$. For $\theta_c < \theta \leq \pi/2$ there is a reflected wave accompanied by an interfacial wave and for $\theta = \theta_c$ the interfacial wave becomes a plane shear (body) wave propagating parallel to the boundary in $x_2 < 0$ (grazing reflection). When $0 < \theta < \theta_c$ the interfacial wave is replaced by a second reflected wave with angle of reflection, θ' say, obtained from (24) by replacing m by $-i\tan\theta'$ to give

$${t'}^2 = \{\lambda^2(\lambda^2 - 2) - t^2\}/(1 + t^2),$$

where $t' = \tan\theta'$.

In $x_2 > 0$, from the counterpart of (10) we see that $m^* = \pm 1$, which yields an interfacial wave in the half-space $x_2 > 0$ for $m^* = +1$. The zeros of the other quadratic factor correspond to $m^* = i\tan\theta^*$ and $m^* = -i\tan\theta^*$, where $m^* = i\tan\theta^*$ corresponds to a transmitted wave provided $\tan\theta^*$ is real and positive. The value of $\tan\theta^*$ is obtained by using the propagation condition (15) and the counterpart of (9) together with Snell's law (22). This gives

$${t^*}^2 = t^2\{t^2 + \lambda^2(2 - \lambda^2)\}/(1 + t^2), \tag{26}$$

and the notation $t^* = \tan\theta^*$ has been introduced.

(26) shows that ${t^*}^2$ is positive when $t^2 > \lambda^2(\lambda^2 - 2) = {t_c}^2$ and negative when $t^2 < {t_c}^2$. Therefore there is a transmitted wave accompanied by an interfacial wave when $\theta \in (\theta_c, \pi/2]$ and for $\theta \in (0, \theta_c)$ the transmitted wave is replaced by a second interfacial

wave. To best of the knowledge of author, the *phenomenon of two interfacial waves* (in the upper half-space), *never appeared in linear elasticity.* In $x_2 > 0$ at $\theta = \theta_c$ there will be grazing transmission.

The coefficients R, R', R^* and $R^{*'}$ are determined by using the boundary conditions (18), and (19), with the second equation in (19) taking the form

$$(2\lambda^2+1)\psi_{,112} - (\lambda^4+2)\psi^*_{,112} + \psi_{,222} - \psi^*_{,222} - \overline{\rho}(\ddot{\psi}_{,2} - \ddot{\psi}^*_{,2}) = 0 \tag{27}$$

in this case, where $\overline{\rho}$ is given by (12).

Substitution of ψ and ψ^* from (20) and (21)(with $m^* = 1$) in (18), first equation in (19) and (27) leads to

$$\begin{aligned} 1 + R + R' &= R^* + R^{*'}, \\ t(1-R) - imR' &= t^*R^* + iR^{*'}, \\ (1+R)(t^2-1) - (1+m^2)R' &= (t^{*2}-1)R^* - 2R^{*'}, \\ (R-1)it(m^2+1) + R'm(t^2-1) &= -R^*it^*(2 - m^2t^2 + t^{*2}) - R^{*'}(t^{*2}-1). \end{aligned} \tag{28}$$

In the latter equation use has been made of (23) and (26) in order to simplify the coefficients. In these equations, for given t, m is obtained from (24) and t^* from (26). The values of R, R', R^*, and $R^{*'}$ are obtained from the solution of (28). The resulting expressions are not given here, but using Mathematica [4], graphical results showing the dependence of $|R|$, $|R'|$, $|R^*|$, and $|R^{*'}|$ on θ and different values of λ are given in Section 4.

4 Numerical Results

Graphical results for a selection of values of λ are given in Figs. (3-6). With reference to the slowness curves (superimposed) in Fig. 1(c), which is the relevant one for the Figs. (3-6)(a), it can be seen that there is one reflected wave, one transmitted wave and two interfacial waves for each possible angle of incidence when $2\beta \neq \alpha + \gamma \, (x_2 < 0)$, $2\beta = \alpha + \gamma \, (x_2 > 0)$. At $\theta = \pi/2$, incident wave is transmitted fully, as $|R| = |R'| = |R^{*'}| = 0$. At grazing incidence ($\theta = 0$), there are no interfacial waves.

In respect of Figs. (3 - 6)(c & d), from the discussion of slowness curves in Fig. 2(c) it is apparent that there are two interfacial waves in $x_2 > 0$ and two reflected waves in $x_2 < 0$ for $0 \leq \theta \leq \theta_c$. A reflected wave with an interfacial wave (in $x_2 < 0$) and a transmitted wave along with an interfacial wave (in $x_2 > 0$) exist for $\theta_c \leq \theta \leq \pi/2$, where the critical angle θ_c is given by (25). For $\lambda = 1.5, 1.6, 1.73,$ and 2.5 the value of $\theta_c = 0.512, 0.962, 1.25,$ and 1.53 respectively. Note the continuity of graphs at $\theta = \theta_c$. Each of $|R|$, $|R^*|$, and $|R^{*'}|$ vanish at one value of θ when $\lambda > \sqrt{2}$.

In Figs. (4-6) the change in the vertical scales must be noted. In addition, an interesting conclusion is the graphs of $|R|$, $|R'|$, $|R^*|$, and $|R^{*'}|$ are varying significantly by a minor change in the stretch λ, as shown in Figs. (3 - 6)(a, b, & c). In general the maximum values of $|R|$, $|R'|$, $|R^*|$, and $|R^{*'}|$ increase as stretch increases.

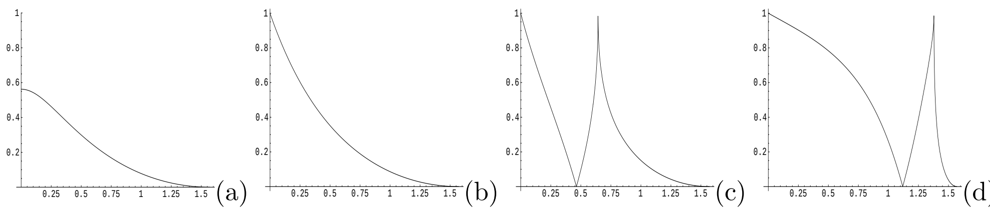

Figure 3: Plots of $|R|$ (in $x_2 < 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d) 2.5.

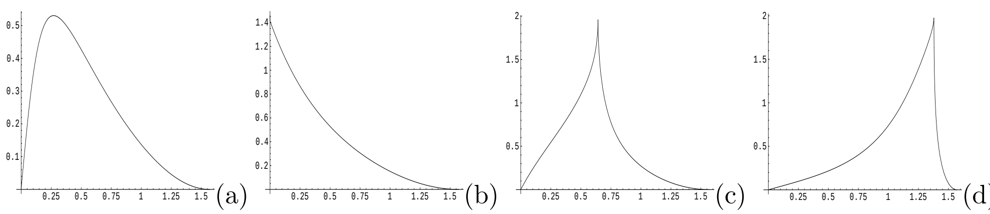

Figure 4: Plots of $|R'|$ (in $x_2 < 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d) 2.5.

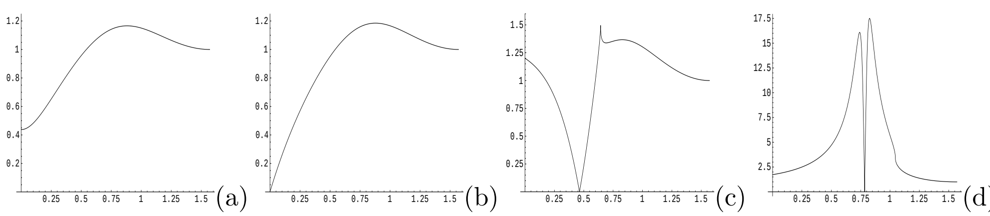

Figure 5: Plots of $|R^*|$ (in $x_2 > 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d)1.73.

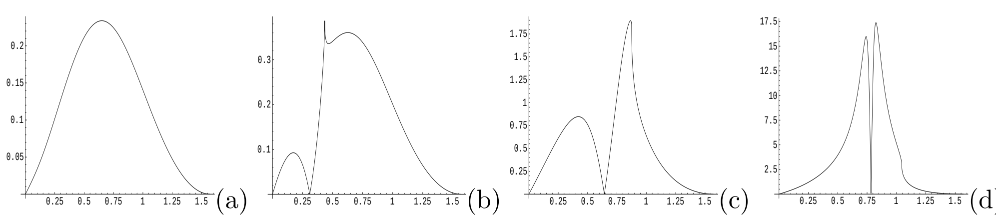

Figure 6: Plots of $|R^{*'}|$ (in $x_2 > 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 1.4, (b) 1.45, (c) 1.6, (d) 1.73.

References

[1] W. Hussain and R. W. Ogden, Reflection and transmision of plane waves at a shear-twin interface, Int. J. Engng Sci., 38 (2000), 1789–1810.

[2] W. Hussain and R. W. Ogden, On the reflection of plane waves at the boundary of an elastic half-space subject to simple shear, Int. J. Engng Sci., 37 (1999), 1549–1576.

[3] R. W. Ogden, Non-linear Elastic Deformations, Dover Publications, Inc. Mineola, New York, 1997.

[4] S. Wolfram, Mathematica, version 5, Wolfrom Research, Champaign, Illinois, 2003.