On The Propagation Of Plane Waves At A Pure Shear Interface*

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Received 12 January 2007

Abstract

Propagation of elastic waves at a pure shear interface of two half-spaces with different strain-energy functions is studied. An interesting result of the existence of two interfacial waves (with no transmitted waves) in the upper half-space is found. The wave amplitudes are illustrated graphically highlighting the dependence on the finite pure shear deformation and the angle of incidence.

1 Introduction

In [1] Hussain and Ogden considered the effect of simple shear deformation on the reflection and transmission of homogeneous plane waves at the boundary between two half-spaces consist of the same material and corresponding to the same strain-energy functions (see also the related paper [2]). In the present paper the effect of pure shear on the reflection and transmission of plane waves at the boundary between two half-spaces corresponding to different strain-energy functions to consist of same incompressible isotropic elastic material is considered.

The required equations and notations are summarized in Section 2 and in the Sub-sections the propagation of plane harmonic waves is discussed with reference to the slowness curves appropriate for the two distinct classes of strain-energy functions.

For the mixed case of the strain-energy functions, the method for finding the amplitudes of the reflected, transmitted and interfacial waves is discussed in Section 3.1. For each angle of incidence a single reflected wave, with angle of reflection equal to the angle of incidence, is generated when a homogeneous plane (SV) wave is incident on the boundary from one half-space, and it is accompanied by an interfacial wave. It is shown that under a certain restriction on the state of deformation a transmitted (homogeneous plane SV) wave and an interfacial wave are generated for all angles of incidence. When this restriction does not hold there exists a critical angle such that (a) in $x_2 < 0$, above which, there is a reflected wave and an interfacial wave but below which there are two reflected waves (and no interfacial waves), and (b) in $x_2 > 0$, ...

*Mathematics Subject Classifications: 74Bxx,74Jxx.
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above which, there is again a transmitted and an interfacial wave but below which there are two interfacial waves (and no transmitted wave).

The theory in Section 3 is illustrated in Section 4 using graphical results to show the dependence of the amplitudes of the waves on the angle of incidence for representative values of the deformation parameters.

2 Basic Equations

Consider an incompressible isotropic elastic material subject to pure shear. Let $\lambda_1$, $\lambda_2$, $\lambda_3$ denote the principal stretches of the deformation. Then, the incompressibility condition is expressed as

$$\lambda_1 \lambda_2 \lambda_3 = 1.$$  

If $\sigma_i (i = 1, 2, 3)$ are the principal Cauchy stresses then the pure shear deformation confined to the (1-2)-plane is given by

$$\lambda_1 = \lambda \neq 1, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1 \text{ with } \sigma_1 \neq 0, \quad \sigma_2 = 0,$$

where a non-vanishing stress $\sigma_3$ is required to maintain $\lambda_3 = 1$.

The equation of motion in terms of a scalar function, $\psi(x_1, x_2, t)$ say, is given by

$$\alpha \psi_{,1111} + 2\beta \psi_{,1122} + \gamma \psi_{,2222} = \rho (\ddot{\psi}_{,11} + \ddot{\psi}_{,22}),$$

where $\alpha, \beta, \gamma$ are defined by

$$\alpha = A_{01212}, \quad \beta = A_{02121}, \quad \gamma = A_{00111} + A_{02222} - 2A_{01122} - 2A_{01221}. \quad (2)$$

In (2) $A_{ijkl}$ are the components of the fourth-order tensor $\mathbf{A}_0$ of instantaneous elastic moduli (see, for example, Ogden [3]).

The shear and normal components of the incremental nominal traction $\Sigma_{21}, \Sigma_{22}$ on a plane $x_2 = \text{constant}$ are expressible in terms of $\psi$ through

$$\Sigma_{21} = \gamma \psi_{,22} - (\gamma - \sigma_2) \psi_{,11},$$

$$-\Sigma_{22,1} = (2\beta + \gamma - \sigma_2) \psi_{,112} + \gamma \psi_{,222} - \rho \ddot{\psi}_{,2}. \quad (3)$$

2.1 Plane Waves

Consider time-harmonic homogeneous plane waves of the form

$$\psi = A \exp[i(k(x_1 \cos \theta + x_2 \sin \theta - ct))],$$

where $A$ is a constant, $c (> 0)$ the wave speed, $k (> 0)$ the wave number and $(\cos \theta, \sin \theta)$ the direction cosines of the direction of propagation of the wave in the $(x_1, x_2)$-plane. Substitution of (4) into (1) gives

$$\alpha \cos^4 \theta + 2\beta \sin^2 \theta \cos^2 \theta + \gamma \sin^4 \theta = \rho c^2. \quad (5)$$
Equation (5) is a relationship between the wave speed and the propagation direction in the \((x_1, x_2)\)-plane and is called the propagation condition. The material constants are taken to satisfy the strong ellipticity inequalities

\[
\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha \gamma},
\]  

(6)

and it is clear from (5) that \(\rho c^2 > 0\) if and only if (6) hold.

Similarly, from (1), for an inhomogeneous plane wave of the form

\[
\psi = \hat{A} \exp[i k' (x_1 - i m x_2 - c' t)],
\]  

(7)

we obtain

\[
\alpha - 2\beta m^2 + \gamma m^4 = \rho (1 - m^2) c'^2,
\]  

(8)

which relates the wave speed \(c'\) to the ‘inhomogeneity factor’ \(m\). Note that the wave decays exponentially as \(x_2 \to -\infty (+\infty)\) provided \(m\) has positive (negative) real part.

Consider two distinct cases corresponding to different strain-energy functions. For these either \(2\beta = \alpha + \gamma\) or \(2\beta \neq \alpha + \gamma\).

### 2.1.1 Case A: \(2\beta = \alpha + \gamma\)

For this case equations (5) and (8) reduce to

\[
\alpha \cos^2 \theta + \gamma \sin^2 \theta = \rho c^2
\]  

(9)

and

\[
(m^2 - 1)(\alpha - \gamma m^2 - \rho c^2) = 0
\]  

(10)

respectively.

In terms of the slowness vector \((s_1, s_2)\) defined by

\[
(s_1, s_2) = (\cos \theta, \sin \theta)/c
\]

equation (9) becomes the slowness curve (by using \(\alpha/\gamma = \lambda^4\))

\[
\lambda^4 s_1^2 + s_2^2 = \overline{\rho},
\]  

(11)

in the \((s_1, s_2)\)-space, where \(\overline{\rho}\) is defined by

\[
\overline{\rho} = \rho/\gamma.
\]  

(12)

By using the dimensionless notation \((\overline{s}_1, \overline{s}_2)\) defined by

\[
(\overline{s}_1, \overline{s}_2) \equiv (s_1, s_2)/\sqrt{\overline{\rho}}.
\]  

(13)

we can write (11) as

\[
\lambda^4 \overline{s}_1^2 + \overline{s}_2^2 = 1.
\]  

(14)
2.1.2 Case B: \(2\beta \neq \alpha + \gamma\)

In this case we take the strain-energy function to satisfy \(\beta = \sqrt{\alpha \gamma}\) which was used by Hussain and Ogden in [1]. Then (5) takes the form

\[
\sqrt{\alpha \cos^2 \theta + \gamma \sin^2 \theta} = pc^2
\]

and (8) becomes

\[
(\sqrt{\alpha} - \sqrt{\gamma}m^2)^2 = \rho(1 - m^2)c'^2.
\]

The slowness curve corresponding to (15) is given by

\[
[\lambda_2 \pi_1^2 + \pi_2^2]^2 = \pi_1^2 + \pi_2^2, \tag{17}
\]

in dimensionless form with the notation (13) and \(\mathbf{p}\) defined by (12). We now show graphically the dependence of the slowness curves on \(\lambda\) for both classes of strain-energy functions in \((\pi_1, \pi_2)\)-space with reference to (14) and (17) (See Figs. 1-2).

![Figure 1](attachment:figure1.png)

Figure 1: Slowness curves in \((\pi_1, \pi_2)\)-space for \(\lambda = 1.4\) with (a) \(2\beta = \alpha + \gamma\), (b) \(2\beta \neq \alpha + \gamma\), (c) the superposition of Figs. in (a) and (b).

![Figure 2](attachment:figure2.png)

Figure 2: Slowness curves in \((\pi_1, \pi_2)\)-space for \(\lambda = 2.5\) with (a) \(2\beta = \alpha + \gamma\), (b) \(2\beta \neq \alpha + \gamma\), (c) the superposition of Figs. in (a) and (b).
3 Reflection and transmission at the interface

Consider the case in which a half-space defined by \( x_2 < 0 \) is joined to a half-space \( x_2 > 0 \). Both are subjected to pure shear deformation. The boundary conditions corresponding to continuous incremental displacement across the interface \( x_2 = 0 \) in terms of the scalar functions \( \psi \) and \( \psi^* \), take the forms

\[
\psi_{1,1} = \psi_{1,1}^*, \; \psi_{2,2} = \psi_{2,2}^*,
\]

respectively on \( x_2 = 0 \), where an asterisk signifies a quantity in \( x_2 > 0 \).

In terms of \( \psi \), boundary conditions corresponding to continuous incremental traction, across the interface \( x_2 = 0 \) are recast through

\[
\Sigma_{21} = \Sigma_{21}^*, \; \Sigma_{22,1} = \Sigma_{22,1}^*,
\]

on use of (3), in the forms

\[
(2\beta + \gamma)(\psi_{112} - \psi_{112}^*) + \gamma(\psi_{222} - \psi_{222}^*) - \rho(\ddot{\psi}_{2} - \ddot{\psi}_{2}^*) = 0.
\]

Because of the symmetry of slowness curves, with respect to the normal direction to the interface, the solution comprising the incident wave, a reflected wave and an interfacial wave in \( x_2 < 0 \) is written as

\[
\psi = A \exp[i(k(x_1 \cos \theta + x_2 \sin \theta - ct))] + AR \exp[i(k(x_1 \cos \theta - x_2 \sin \theta - ct))] + AR' \exp[i(k'(x_1 - imx_2 - c't))],
\]

where \( R \) is the reflection coefficient and \( R' \) measures the amplitude of the interfacial wave. The notations \( k', m, c' \) are as used in (7) and \( m \) has positive real part.

Accordingly, in \( x_2 > 0 \), the wave solution may be written

\[
\psi^* = AR^* \exp[i(k^*(x_1 \cos \theta^* + x_2 \sin \theta^* - c^*t))] + AR'^* \exp[i(k'^*(x_1 + im^*x_2 - c'^*t))],
\]

comprising a transmitted and an interfacial wave, where \( R^* \) is the transmission coefficient and \( R'^* \) is the analogue of \( R' \) for \( x_2 > 0 \). Note that the interfacial wave decays as \( x_2 \to \infty \) provided \( m^* \) has positive real part.

Here Snell’s law takes the form

\[
\cos \theta/c = 1/c' = \cos \theta^*/c^* = 1/c'^*.
\]

(22) states in particular, that the first components of the slowness vectors for each homogeneous plane wave interacting at the boundary \( x_2 = 0 \) are equal.

Thus, by reference to the slowness curves (superimposed) as exemplified in Fig. 1(c) and Fig. 2(c), the range of angles of incidence for which a transmitted wave exists can be identified. In Fig. 1(c), for example, if the outer curve corresponds to \( x_2 < 0 \) there is, for every angle of incidence (i.e. for every \( s_1 \) associated with the curve) a point on the inner curve (corresponding to \( x_2 > 0 \)), and hence a transmitted wave.
In Fig. 2(c) on the other hand, there are values of \( s_1 \) on the slowness (outer) curve for \( x_2 < 0 \) for which there are no corresponding values on the inner slowness curve (corresponding to \( x_2 > 0 \)), and therefore a range of angles of incidence for which no transmitted wave exists. This will be discussed further in Section 3.1.

We now examine here the case in which \( 2\beta \neq \alpha + \gamma \) (\( x_2 < 0 \)), \( 2\beta = \alpha + \gamma \) (\( x_2 > 0 \)).

### 3.1 \( 2\beta \neq \alpha + \gamma \) (\( x_2 < 0 \)), \( 2\beta = \alpha + \gamma \) (\( x_2 > 0 \))

In this case we see from (16), after using Snell’s law \( \cos \beta/c = 1/c' \), we have

\[
(m^2 + t^2)[m^2(1 + t^2) - t^2 + \lambda^2(\lambda^2 - 2)] = 0,
\]

where \( t = \tan \theta \). Note that \( t \) should be distinguished from the time variable \( t \) used earlier. \( m = +it \) and \( m = -it \) are the solutions of (23) corresponding to the incident and reflected waves respectively. The other solutions are

\[
m = \pm \sqrt{1 - (\lambda^2 - 1)^2/(1 + t^2)}.
\]

If \( \lambda \leq \sqrt{2} \) the \( m \) is real for all \( \theta \) and the positive solution of (24) corresponds to an interfacial wave in \( x_2 < 0 \). If \( \lambda > \sqrt{2} \) then there is a critical value of \( \theta, \theta_c \) say, for which \( m = 0 \) and this is given by

\[
t^2_c = \lambda^2(\lambda^2 - 2),
\]

where the notation \( t_c = \tan \theta_c \) is used. It follows that \( m \) is real for \( \theta_c \leq \theta \leq \pi/2 \). For \( \theta_c < \theta \leq \pi/2 \) there is a reflected wave accompanied by an interfacial wave and for \( \theta = \theta_c \) the interfacial wave becomes a plane shear (body) wave propagating parallel to the boundary in \( x_2 < 0 \) (grazing reflection). When \( 0 < \theta < \theta_c \) the interfacial wave is replaced by a second reflected wave with angle of reflection, \( \theta' \) say, obtained from (24) by replacing \( m \) by \(-i\tan \theta'\) to give

\[
t^2' = \{\lambda^2(\lambda^2 - 2) - t^2\}/(1 + t^2),
\]

where \( t' = \tan \theta' \).

In \( x_2 > 0 \), from the counterpart of (10) we see that \( m^* = \pm 1 \), which yields an interfacial wave in the half-space \( x_2 > 0 \) for \( m^* = +1 \). The zeros of the other quadratic factor correspond to \( m^* = i\tan \theta^* \) and \( m^* = -i\tan \theta^* \), where \( m^* = i\tan \theta^* \) corresponds to a transmitted wave provided \( \tan \theta^* \) is real and positive. The value of \( \tan \theta^* \) is obtained by using the propagation condition (15) and the counterpart of (9) together with Snell’s law (22). This gives

\[
t^{*^2} = t^2\{t^2 + \lambda^2(2 - \lambda^2)\}/(1 + t^2),
\]

and the notation \( t^* = \tan \theta^* \) has been introduced.

(26) shows that \( t^2 \) is positive when \( t^2 > \lambda^2(\lambda^2 - 2) = t_c^2 \) and negative when \( t^2 < t_c^2 \). Therefore there is a transmitted wave accompanied by an interfacial wave when \( \theta \in (\theta_c, \pi/2) \) and for \( \theta \in (0, \theta_c) \) the transmitted wave is replaced by a second interfacial
wave. To best of the knowledge of author, the phenomenon of two interfacial waves (in the upper half-space), never appeared in linear elasticity. In $x_2 > 0$ at $\theta = \theta_c$ there will be grazing transmission.

The coefficients $R, R', R^*$ and $R'^*$ are determined by using the boundary conditions (18), and (19), with the second equation in (19) taking the form

$$ (2\lambda^2 + 1)\psi_{112} - (\lambda^4 + 2)\psi_{222}^* + \psi_{222}' - \psi_{222}^* - \overline{\rho}(\psi_{12}' - \psi_{22}^*) = 0 $$

(27)

in this case, where $\overline{\rho}$ is given by (12).

Substitution of $\psi$ and $\psi^*$ from (20) and (21)(with $m^* = 1$) in (18), first equation in (19) and (27) leads to

$$ 1 + R + R' = R^* + R'^*, $$
$$ t(1 - R) - imR' = t^*R^* + iR'^*, $$
$$ (1 + R)(t^2 - 1) - (1 + m^2)R' = (t^2 - 1)R^* - 2R'^*, $$
$$ (R - 1)it(m^2 + 1) + R'm(t^2 - 1) = -R^*it^*(2 - m^2t^2 + t^{*2}) - R'^*(t^2 - 1). $$

(28)

In the latter equation use has been made of (23) and (26) in order to simplify the coefficients. In these equations, for given $t, m$ is obtained from (24) and $t^*$ from (26). The values of $R, R', R^*$, and $R'^*$ are obtained from the solution of (28). The resulting expressions are not given here, but using Mathematica [4], graphical results showing the dependence of $|R|, |R'|, |R^*|$, and $|R'^*|$ on $\theta$ and different values of $\lambda$ are given in Section 4.

### 4 Numerical Results

Graphical results for a selection of values of $\lambda$ are given in Figs. (3-6). With reference to the slowness curves (superimposed) in Fig. 1(c), which is the relevant one for the Figs. (3-6)(a), it can be seen that there is one reflected wave, one transmitted wave and two interfacial waves for each possible angle of incidence when $2\beta \neq \alpha + \gamma (x_2 < 0), 2\beta = \alpha + \gamma (x_2 > 0)$. At $\theta = \pi/2$, incident wave is transmitted fully, as $|R| = |R'| = |R^*| = 0$. At grazing incidence ($\theta = 0$), there are no interfacial waves.

In respect of Figs. (3-6)(c & d), from the discussion of slowness curves in Fig. 2(c) it is apparent that there are two interfacial waves in $x_2 > 0$ and two reflected waves in $x_2 < 0$ for $0 \leq \theta \leq \theta_c$. A reflected wave with an interfacial wave (in $x_2 < 0$) and a transmitted wave along with an interfacial wave (in $x_2 > 0$) exist for $\theta_c \leq \theta \leq \pi/2$, where the critical angle $\theta_c$ is given by (25). For $\lambda = 1.5, 1.6, 1.73,$ and 2.5 the value of $\theta_c = 0.512, 0.962, 1.25,$ and 1.53 respectively. Note the continuity of graphs at $\theta = \theta_c$. Each of $|R|, |R'|, |R^*|$, and $|R'^*|$ vanish at one value of $\theta$ when $\lambda > \sqrt{2}$.

In Figs. (4-6) the change in the vertical scales must be noted. In addition, an interesting conclusion is the graphs of $|R|, |R'|, |R^*|$, and $|R'^*|$ are varying significantly by a minor change in the stretch $\lambda$, as shown in Figs. (3-6)(a, b, & c). In general the maximum values of $|R|, |R'|, |R^*|$, and $|R'^*|$ increase as stretch increases.
Figure 3: Plots of $|R|$ (in $x_2 < 0$) against $\theta$ ($0 \leq \theta \leq \pi/2$) with the following values of $\lambda$: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d) 2.5.

Figure 4: Plots of $|R'|$ (in $x_2 < 0$) against $\theta$ ($0 \leq \theta \leq \pi/2$) with the following values of $\lambda$: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d) 2.5.

Figure 5: Plots of $|R^*|$ (in $x_2 > 0$) against $\theta$ ($0 \leq \theta \leq \pi/2$) with the following values of $\lambda$: (a) 1.4, (b) $\sqrt{2}$, (c) 1.5, (d) 1.73.

Figure 6: Plots of $|R'|$ (in $x_2 > 0$) against $\theta$ ($0 \leq \theta \leq \pi/2$) with the following values of $\lambda$: (a) 1.4, (b) 1.45, (c) 1.6, (d) 1.73.


References


