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Convexities Of Some Functions Involving The Polygamma Functions*

Jian Cao†, Da-Wei Niu‡, Feng Qi§

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Abstract

In this short note, the convexity and concavity of two functions related to the polygamma functions are discussed.

1 Introduction

It is well known that the psi or digamma function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function Γ, that is, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and that $\psi^{(i)}$ for $i \in \mathbf{N}$ are called polygamma functions. It is also well known that the following representations [1] are valid:

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{v=1}^\infty \frac{x}{v(v+x)}, \tag{1}$$

$$(-1)^{n+1}\psi^{(n)}(x) = \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}} dt = n! \sum_{v=0}^\infty \frac{1}{(v+x)^{n+1}}, \tag{2}$$

where $n \in \mathbf{N}$ and $\gamma = 0.57721\dots$ is Euler-Mascheroni's constant.

The psi and polygamma functions play a central role in the theory of special functions, they have many important applications in different branches such as mathematical physics and statistics. For more information on the psi and polygamma functions, please refer to [3, 4, 5, 6, 7, 9] and the references therein.

In 2004, it was proved in [6] that the function $\psi(e^x)$ is strictly concave on $\mathbf{R}$ and the function $\psi(x^c)$ is strictly concave (convex, respectively) on $(0, \infty)$ if and only if $c > 0$ ($c \in [-1, 0)$, respectively).

Now it is natural to ask for the convexity or concavity of two more general functions $F(k, x) = \psi^{(k)}(e^x)$ for $x \in \mathbf{R}$ and $G(k, x) = \psi^{(k)}(x^c)$ for $x > 0$. The aim of this short

*Mathematics Subject Classifications: 26D15, 33B15.

†Department of Mathematics, East China Normal University, Shanghai City, 200241, China

‡College of Information and Business, Zhongyuan University of Technology, Zhengzhou City, Henan Province, 450007, China

§Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

paper is to study these properties of $F(k,x)$ and $G(k,x)$. The main results are as follows.

THEOREM 1. The function $F(k,x)=\psi^{(k)}(e^x)$ is concave (convex, respectively) on $\mathbf{R}$ if $k=2n-2$ ($k=2n-1$, respectively), where $n\in\mathbf{N}$.

REMARK 1. By (1), it is easy to see that $F(2n-2,x)$ is increasing and $F(2n-1,x)$ is decreasing for $n\in\mathbf{N}$.

THEOREM 2. Let $c\neq 0$ be a real number and $n\in\mathbf{N}$. Then the function $G(k,x)=\psi^{(k)}(x^c)$ is convex in $(0,\infty)$ if either $k=2n-1$ and $c\in\left(-\infty,-\frac{1}{2n-1}\right]\cup(0,\infty)$ or $k=2n-2$ and $c\in[-\frac{1}{2n-1},0)$, and it is concave if either $k=2n-1$ and $c\in[-\frac{1}{2n},0)$, or $k=2n-2$ for $n\geq 2$ and $c\in\left(-\infty,-\frac{1}{2n-2}\right]\cup(0,\infty)$, or $n=1$ and $c\in(0,\infty)$.

REMARK 2. Letting $k=0$ in Theorem 1 and Theorem 2 leads to the corresponding results obtained in [3].

2 Lemmas

To prove our main results, the following lemmas are necessary.

LEMMA 1. The following formulas [1, p. 255] are valid:

$$\psi^{(n)}(x+1) = \psi^{(n)}(x)+(-1)^n\frac{n!}{x^{n+1}} \quad \text{for } x>0 \text{ and } n=0,1,2,..., \tag{3}$$

$$\psi(x) \sim \ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\cdots \quad \text{as } x\to\infty, \tag{4}$$

$$\left|\psi^{(n)}(x)\right| \sim \frac{(n-1)!}{x^n}+\frac{n!}{2x^{n+1}}+\frac{(n+1)!}{12x^{n+2}}-\cdots \text{ as } x\to\infty \text{ for } n=1,2,\ldots. \tag{5}$$

LEMMA 2. Let $n\in\mathbf{N}$. Then the function $\frac{x\psi^{(n+1)}(x)}{\psi^{(n)}(x)}$ is strictly increasing from $[0,\infty)$ onto $[-(n+1),-n)$.

PROOF. A proof for the monotonicity of $f(x)=x\psi^{(n+1)}(x)/\psi^{(n)}(x)$ has given in [4]. Using (3) and (5), it is easy to conclude that $\lim_{x\to 0^+}f(x)=-(n+1)$ and $\lim_{x\to\infty}f(x)=-n$.

3 Proofs of Theorems

PROOF OF THEOREM 1. Let $x\in\mathbf{R}$. Differentiation gives:

$$e^{-2x}F''(k,x)=\frac{1}{z}\psi^{(k+1)}(z)+\psi^{(k+2)}(z), \tag{6}$$

where $z=e^x$. Applying the integral representations in (1) and

$$\frac{k!}{x^{k+1}}=\int_0^\infty e^{-xt}t^k dt \tag{7}$$

for $x > 0$ and $k = 0, 1, 2, \ldots$ and making use of the convolution theorem for Laplace transforms (see [11]) leads to

$$e^{-2x}F''(k,x) = \int_0^\infty e^{-zt}\Delta(k,t)dt, \tag{8}$$

where

$$\Delta(k,t) = \begin{cases} \displaystyle\int_0^t \frac{s^{2n-1}}{1-e^{-s}}ds - \frac{t^{2n}}{1-e^{-t}}, & \text{for } k = 2n-2; \\ \displaystyle -\int_0^t \frac{s^{2n}}{1-e^{-s}}ds + \frac{t^{2n+1}}{1-e^{-t}}, & \text{for } k = 2n-1. \end{cases} \tag{9}$$

If $k = 2n - 2$, $n \in \mathbf{N}$, direct calculation yields

$$\begin{gathered} (1-e^{-t})^2 t^{1-2n} e^t \Delta'(k,t) = e^t - 1 + t - 2ne^t + 2n \triangleq f(t), \\ f'(t) = e^t + 1 - 2ne^t < 0. \end{gathered} \tag{10}$$

Since $\lim_{t\to 0^+} f(t) = 0$ and $\lim_{t\to 0^+} \Delta(k,t) = 0$, thus $F''(k,x) < 0$.

If $k = 2n - 1$, straightforward computation shows

$$\begin{gathered} (1-e^{-t})^2 t^{-2n} e^t \Delta'(k,t) = 2ne^t - 2n - t \triangleq g(t), \\ g'(t) = 2ne^t - 1 > 0. \end{gathered} \tag{11}$$

Because of $\lim_{t\to 0^+} g(t) = 0$ and $\lim_{t\to 0^+} \Delta(k,t) = 0$, therefore $F''(k,x) > 0$. The proof of Theorem 1 is complete.

PROOF OF THEOREM 2. Let $x > 0$ and $z = x^c > 0$. Then

$$\begin{aligned} \frac{\partial^2 G(k,x)}{\partial x^2} &= (-1)^k c^2 x^{c-2} \psi^{(k+1)}(z) \left[(-1)^k z \frac{\psi^{(k+2)}(z)}{\psi^{(k+1)}(z)} + (-1)^k \frac{c-1}{c} \right] \\ &\triangleq (-1)^k c^2 x^{c-2} \psi^{(k+1)}(z) h(k,z). \end{aligned} \tag{12}$$

From (1), we can see that $G(k,x)$ is concave (convex, respectively) if $h(k,z) < 0$ ($h(k,z) > 0$, respectively).

Utilizing Lemma 2 yields for $z > 0$ the following conclusions: For $k = 2n - 2$,

$$-2n + 1 - \frac{1}{c} < h(k,z) = z\frac{\psi^{(2n)}(z)}{\psi^{(2n-1)}(z)} + \frac{c-1}{c} < -2n + 2 - \frac{1}{c}. \tag{13}$$

Thus, if $c \in \left[-\frac{1}{2n-1}, 0\right)$, then $\frac{\partial^2 G(k,x)}{\partial x^2} > 0$; if either $n \geq 2$ and $c \in \left(-\infty, -\frac{1}{2n-2}\right] \cup (0, \infty)$ or $n = 1$ and $c \in (0, \infty)$, then $\frac{\partial^2 G(k,x)}{\partial x^2} < 0$. For $k = 2n - 1$,

$$2n - 1 + \frac{1}{c} < h(k,z) = -z\frac{\psi^{(2n+1)}(z)}{\psi^{(2n)}(z)} - \frac{c-1}{c} < 2n + \frac{1}{c}. \tag{14}$$

As a result, if $c \in \left(-\infty, -\frac{1}{2n-1}\right] \cup (0, \infty)$, then $\frac{\partial^2 G(k,x)}{\partial x^2} > 0$; if $c \in \left[-\frac{1}{2n}, 0\right)$, then $\frac{\partial^2 G(k,x)}{\partial x^2} < 0$. The proof of Theorem 2 is complete.

4 Remarks

It is well-known that some inequalities for the beta and gamma functions can be obtained from classical inequalities, see [10] and [2, 8]. In [8], it was proved that the digamma or psi function is nondecreasing and concave on $(0,\infty)$ and

$$\psi(ax+by)\ge a\psi(x)+b\psi(y),\quad x,y>0,\quad a,b\ge 0,\quad a+b=1. \tag{15}$$

Theorem 1 implies for $k=2n-2$, $x_1,x_2\in\mathbf{R}$ and $a+b=1$ with $a>0$ and $b>0$ that

$$a\psi^{(2n-2)}(e^{x_1})+b\psi^{(2n-2)}(e^{x_2})\le \psi^{(2n-2)}(e^{ax_1+bx_2}). \tag{16}$$

Since the function e^x is convex, thus

$$e^{ax_1+bx_2}\le ae^{x_1}+be^{x_2}. \tag{17}$$

Let $x_1=\ln x$ and $x_2=\ln y$. Then inequality (16) can be rewritten as

$$a\psi^{(2n-2)}(x)+b\psi^{(2n-2)}(y)\le \psi^{(2n-2)}(x^ay^b)\le \psi^{(2n-2)}(ax+by). \tag{18}$$

If $k=2n-1$, the double inequality (18) reverses.

If taking $n=1$ in the double inequality (18), then inequality (15) can be deduced.

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