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A Weak Regular Splitting For Singular Linear Systems*

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Abstract

In this paper, the semiconvergence of a proper weak regular splitting method for the singular linear algebraic system $Ax = b$ with a singular M-matrix A is discussed. The extrapolated iterative method based on the weak regular splitting iterative method is also constructed.

Let us consider the solution of a linear system of equations

$$Ax = b \tag{1}$$

by iterative methods. Here A is an $n \times n$ singular matrix, $x, b \in R^n$ with b known and x unknown. We assume that the system (1) is solvable, i.e., it has at least a solution, which is equivalent to $b \in R(A)$, where $R(A)$ denotes the range of A.

In order to solve the system (1) with iterative methods usually we construct the iteration of the form

$$x^0 \in R^n; \quad x^{k+1} = Tx^k + c, \quad k = 0, 1, 2, \dots . \tag{2}$$

Here the question is the construction of the iterative matrix T and the vector c. The usual approach is through the splitting of the coefficient matrix A:

$$A = M - N,$$

by the formulas

$$T = M^{-1}N, \quad c = M^{-1}b,$$

where M is nonsingular.

It is well known that for singular system the iterative method (2) is semiconvergent if, and only if, the associated convergence factor

$$\upsilon(T) \equiv \max\{|\lambda|, \lambda \in \sigma(T)\backslash\{1\}\} < 1, \tag{3}$$

and the elementary divisors associated with $\mu = 1 \in \sigma(T)$ are linear, i.e.,

$$\text{index}(I - T) = 1, \tag{4}$$

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where $\upsilon(T)$ denotes the pesudo-spectral radius of T, $\sigma(T)$ denotes the spectrum of T and index(B) denotes the index of the matrix B, i.e., the smallest nonnegative integer k such that $\mathrm{rank}(B^k) = \mathrm{rank}(B^{k+1})$. In this case, T is called a semiconvergent matrix.

DEFINITION 1 [1]. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called a singular M-matrix if A can be expressed in the form

$$A = sI - B, \quad s > 0, \quad B \geqslant 0, \tag{5}$$

and $s = \rho(B)$, where $\rho(B)$ denotes the spectral radius of B.

A singular M-matrix A is said to have "property c" (cf. [1]) if it can be splitted into (5) and the matrix $T = B/s$ is semi-convergent.

DEFINITION 2 [1]. Let $A \in R^{n\times n}$. The splitting $A = M - N$ is called weak regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$.

In [4], the proper weak regular splitting for nonsingular M-matrix was constructed. In the following we prove that for any irreducible singular M-matrix $A \in R^{n\times n}$, there exists an efficient weak regular splitting.

Without loss of generality we can partition A into

$$A = (a_{ij}) = \begin{bmatrix} A_1 & a \\ b^T & a_{nn} \end{bmatrix},$$

where A_1 is a nonsingular M-matrix of order $n-1$, $a \leqslant 0$, $b \leqslant 0$ are in R^{n-1} and $a_{nn} = b^T A_1^{-1} a$. There exist lower and upper triangular nonsingular M-matrices L_1 and U_1, respectively, such that

$$A_1 = L_1 U_1.$$

Set

$$L = (l_{ij}) = \begin{bmatrix} L_1 & 0 \\ b^T U_1^{-1} & 1 \end{bmatrix} \quad \text{and} \quad U = (u_{ij}) = \begin{bmatrix} U_1 & L_1^{-1} a \\ 0 & 0 \end{bmatrix}.$$

We need the following lemma.

LEMMA 1. Let A be an irreducible singular M-matrix and the splitting $A = M - N$ be a weak regular. Then it holds

$$\rho(M^{-1}N) = 1, \quad \mathrm{index}(I - M^{-1}N) = 1.$$

Indeed, there exists a vector $x > 0$ such that $Ax \geqslant 0$ since A is an irreducible singular M-matrix. By [3, Theorem 6], the result is proved.

Now, we set

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n), \ \ 0 < \lambda_i < 1, \ \ i = 1, 2, ..., n-1, \ \ \lambda_n > 0$$

and introduce the notation

$$D = \mathrm{diag}(d_1, d_2, ..., d_n),$$

where

$$d_i = \frac{1 - \lambda_i}{u_{ii}} > 0, \quad i = 1, ..., n-1, \quad d_n = \lambda_n.$$

THEOREM 1. The matrices

$$M = LD^{-1}, \quad N = M - A \tag{6}$$

define a weak regular splitting of the matrix A.

PROOF. Since L is a nonsingular M-matrix, $M^{-1} = DL^{-1} \geqslant 0$. $N = L(D^{-1} - U)$. Therefore we have

$$M^{-1}N = DL^{-1}L(D^{-1} - U) = I - DU.$$

The matrix DU is an upper triangular singular M-matrix with the diagonal entries $1 - \lambda_i$, $i = 1, 2, ..., n - 1$, and $\lambda_n = 0$, which proves the Theorem.

THEOREM 2. Assume that the numbers λ_i satisfy

$$0 < \lambda_i < 1, \quad i = 1, 2, ..., n - 1, \ \lambda_n > 0.$$

Then the iterative method induced by the weak regular splitting of the form (6) is semiconvergent.

PROOF. By the proof of Theorem 1 we know the diagonal entries of the matrix DU are $1-\lambda_i$ $(i = 1, 2, \cdots, n-1)$ and 0. Clearly, all eigenvalues of the iterative matrix $M^{-1}N$ are λ_i $(i = 1, 2, \cdots, n - 1)$ and 1. Therefore $\upsilon(T) = \max |\lambda_i| < 1$. By Lemma 1, $M^{-1}N$ is semiconvergent. The proof is complete.

The extrapolated iterative method based on the stationary iterative method (2) is as follows:

$$x^0 \in R^n; \quad x^{k+1} = (1 - \omega)x^k + \omega(Tx^k + c), \quad k = 0, 1, 2, \dots, \tag{7}$$

where the parameter $\omega \in R\backslash\{0\}$.

Several papers (cf. [5,6,7]) studied the extrapolated iterative methods for singular linear systems. In the following we study the extrapolated iterative methods for singular linear systems base on the weak regular splitting iterative method (6).

THEOREM 3. The extrapolated iterative method (7) based on the weak regular splitting iterative method (6) converges if, and only if, the following conditions are satisfied

(a) $\mathrm{index}(I - T) = 1$;

(b) $0 < \lambda_i < 1$, $\lambda_n > 0$, and $0 < \omega < \frac{2}{1-\min \lambda_i}$, $(i = 1, 2, ..., n - 1)$.

PROOF. Denote the extrapolated iteration matrix of (7) by

$$T_\omega = \omega T + (1 - \omega)I = I - \omega(I - T) = I - \omega DU. \tag{8}$$

Then

$$I - T_\omega = \omega(I - T),$$

from which we have, obviously,

$$\mathrm{index}(I - T_\omega) = \mathrm{index}(I - T). \tag{9}$$

The equality (8) implies

$$\nu(T_\omega) = \max |1 - \omega(1 - \lambda_i)|, \quad i = 1, 2, ..., n - 1. \tag{10}$$

From (10) we deduce that $|1 - \omega(1 - \lambda_i)| < 1$ if, and only if,

$$0 < \omega < \frac{2}{1 - \min \lambda_i}, \quad i = 1, 2, ..., n - 1. \tag{11}$$

According to (9) and (11), Theorem 3 follows due to (3) and (4). The proof is complete.

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