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Sufficient Conditions For Univalence*

Narayanasamy Seenivasagan†

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Abstract

In this paper, for the integral operator $\left\{\beta \int_0^z \xi^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(\xi)}{\xi}\right)^{\frac{1}{\alpha}} d\xi\right\}^{\frac{1}{\xi}}$ to be univalent in the open unit disk, conditions on β, α and $f_i(z)$ are determined.

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{A}_n$ be the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}_1^* := \mathbb{N} \setminus \{0, 1\} = \{2, 3, ...\}). \tag{1}$$

Let T be the univalent [7] subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1 \quad (z \in U).$$

Let T_n be the subclass of T for which $f^{(k)}(0) = 0$ $(k = 2, 3, ..., n)$. Let $T_{n,\mu}$ be the subclass of T_n consisting of functions of the form (1) satisfying

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu \quad (z \in U) \tag{2}$$

for some μ $(0 < \mu \leq 1)$ and let us denote $T_{n,1} \equiv T_n$. Let $\mathcal{S}(p)$ be a subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \quad (z \in U) \tag{3}$$

for some real p with $(0 < p \leq 2)$. Singh [6] has shown that if $f \in \mathcal{S}(p)$, then $f(z)$ satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p|z|^2 \quad (z \in U).$$

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†School of Mathematical Sciences, Universiti Sains Malaysia, USM, Pinang 11800, Malaysia

A subclass $\mathcal{S}_n(p)$ of $\mathcal{A}$ is defined here for which $f \in \mathcal{A}_n$ satisfies (3) and

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p|z|^n \quad (z \in U,\ n \in \mathbb{N}_1^*), \tag{4}$$

and let us denotes $\mathcal{S}_2(p) \equiv \mathcal{S}(p)$. Pascu [3] has proved the following theorem:

THEOREM A. [3, 4] Let $\beta \in \mathbb{C}$, $\Re\beta \geq \gamma > 0$. If $f \in \mathcal{A}$ satisfies

$$\frac{1-|z|^{2\gamma}}{\gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in U),$$

then the integral operator

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) dt \right]^{\frac{1}{\beta}}$$

is in $\mathcal{S}$.

Pescar [5] has obtained the following theorem.

THEOREM B. [5] Let $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha \geq \frac{3}{|\alpha|}$. If $f \in \mathcal{A}$ satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in U)$$

and $|f(z)| \leq 1 \quad (z \in U)$, then the integral operator

$$H_{\alpha,\beta}(z) := \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

is in $\mathcal{S}$.

Using Theorems A and B, Breaz and Owa [2] obtained the following Theorems C, D, and E.

THEOREM C. [2] Let $M \geq 1$, $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha > \frac{(2M+1)k}{|\alpha|}$. Let $f_i \in T_2$ and

$$f_i(z) = z + \sum_{s=3}^{\infty} a_s^i z^s \tag{5}$$

for all $i = 1, 2, ..., k$, $k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\} = \{1, 2, ...\}$ and if

$$|f_i(z)| \leq M \quad (z \in U, \quad i = 1, 2, ..., k),$$

then the integral operator

$$F_{\alpha,\beta}(z) := \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in \mathcal{S} \tag{6}$$

THEOREM D. [2] Let $M \geq 1$, $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha > \frac{k((\mu+1)M+1)}{|\alpha|}$. Let $f_i \in T_{2,\mu}$ be defined by (5) for all $i = 1, 2, ..., k, n \in \mathbb{N}^*$. If $|f_i(z)| \leq M \quad (z \in U, \quad i = 1, 2, ..., k)$, then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

THEOREM E. [2] Let $M \geq 1$, $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha > \frac{k((p+1)M+1)}{|\alpha|}$. Let $f_i \in \mathcal{S}(p)$ be defined by (5) for all $i = 1, 2, ..., k, n \in \mathbb{N}^*$. If $|f_i(z)| \leq M \quad (z \in U, \quad i = 1, 2, ..., k)$, then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

When $M = 1$, Theorems C – E reduce to main results of Breaz and Breaz [1].

Theorem B is true even if $\Re\beta \geq \Re\alpha \geq 3/|\alpha|$ is replaced by the condition $\Re\beta \geq 3/|\alpha|$. Similarly Theorem C is true even if $\Re\beta \geq \Re\alpha \geq \frac{(2M+1)k}{|\alpha|}$ is replaced by the condition $\Re\beta \geq \frac{(2M+1)k}{|\alpha|}$. Theorem D is true even if $\Re\beta \geq \Re\alpha \geq \frac{k((\mu+1)M+1)}{|\alpha|}$ is replaced by the condition $\Re\beta \geq \frac{k((\mu+1)M+1)}{|\alpha|}$ and Theorem E is true even if $\Re\beta \geq \Re\alpha \geq \frac{k((p+1)M+1)}{|\alpha|}$ is replaced by the condition $\Re\beta \geq \frac{k((p+1)M+1)}{|\alpha|}$.

In this paper, Theorems C – E are extended to obtain sufficient conditions for univalence of certain integral operator.

In order to prove the main result of this paper, the following lemma is required.

LEMMA. (Schwarz's Lemma) If the function $w(z)$ is analytic in the unit disk U, $w(0) = 0$, and $|w(z)| \leq 1$, for all $z \in U$, then

$$|w(z)| \leq |z| \quad (z \in U)$$

and equality holds only if $w(z) = \epsilon z$, where $|\epsilon| = 1$.

THEOREM 1. Let $f_i \in T_{n,\mu_i}$ $(i = 1, 2, ..., k,\ k \in \mathbb{N}^*)$ be defined by

$$f_i(z) = z + \sum_{s=n+1}^{\infty} a_s^i z^s \quad (n \in \mathbb{N}_1^*) \tag{7}$$

for all $i = 1, 2, ..., k$, $\alpha, \beta \in \mathbb{C}$, $\Re\beta \geq \gamma$ and

$$\gamma := \sum_{i=1}^{k} \frac{1 + (1 + \mu_i)M}{|\alpha|} \quad (M \geq 1,\ 0 < \mu_i \leq 1,\ k \in \mathbb{N}^*). \tag{8}$$

If

$$|f_i(z)| \leq M \quad (z \in U, \quad i = 1, 2, ..., k),$$

then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

PROOF. Define the function $h(z)$ in U by

$$h(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(\xi)}{\xi}\right)^{\frac{1}{\alpha}} d\xi, \tag{9}$$

then $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \prod_{i=1}^{k} \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha}}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{k} \frac{1}{\alpha}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{10}$$

Equation (10), yields

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{k} \frac{1}{|\alpha|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) = \sum_{i=1}^{k} \frac{1}{|\alpha|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right|\left|\frac{f_i(z)}{z}\right| + 1\right). \tag{11}$$

The hypothesis will then yield $|f_i(z)| \leq M \quad (M \geq 1,\ z \in U, \quad i = 1, 2, ..., k,\ k \in \mathbb{N}^*)$. By Schwarz Lemma, one can obtain that

$$|f_i(z)| \leq M|z| \quad (z \in U, \quad i = 1, 2, ..., k,\ k \in \mathbb{N}^*). \tag{12}$$

Equations (11) and (12), imply

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{k} \frac{1}{|\alpha|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M + 1\right) \leq \sum_{i=1}^{k} \frac{1}{|\alpha|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M + M + 1\right). \tag{13}$$

Since $f_i \in T_{n,\mu_i}$, in view of (8), using (2), (13) may be written as

$$\left|\frac{zh''(z)}{h'(z)}\right| < \sum_{i=1}^{k} \frac{1 + (\mu_i + 1)M}{|\alpha|} = \gamma. \tag{14}$$

On multiplying (14) by $(1 - |z|^{2\gamma})/\gamma$, the following inequality is obtained

$$\frac{1 - |z|^{2\gamma}}{\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 - |z|^{2\gamma} < 1 \quad (z \in U).$$

Since $\Re\beta \geq \gamma > 0$, it follows from Theorem A that

$$\left[\beta \int_0^z \xi^{\beta-1} h'(\xi) d\xi\right]^{\frac{1}{\beta}} \in \mathcal{S}.$$

Since

$$\left[\beta \int_0^z \xi^{\beta-1} h'(\xi) d\xi\right]^{\frac{1}{\beta}} = \left[\beta \int_0^z \xi^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(\xi)}{\xi}\right)^{\frac{1}{\alpha}} d\xi\right]^{\frac{1}{\beta}} = F_{\alpha,\beta}(z),$$

hence $F_{\alpha,\beta}(z) \in \mathcal{S}$.

REMARK 1. By taking $n = 2$ and $\mu_i = \mu$, Theorem 1 is reduced to Theorem D. By taking $\mu_i = \mu = 1$, and $n = 2$, Theorem 1 is reduced to Theorem C.

THEOREM 2. Let $f_i \in \mathcal{S}_n(p)$ $(i = 1, 2, ..., k,\ k \in \mathbb{N}^*,\ n \in \mathbb{N}_1^*)$ defined by (7), $\alpha, \beta \in \mathbb{C}$, $\Re\beta \geq \gamma_1$ and

$$\gamma_1 := \frac{k(1 + (p + 1)M)}{|\alpha|} \quad (M \geq 1). \tag{15}$$

If

$$|f_i(z)| \le M \quad (z \in U, \quad i = 1, 2, ..., k, \ k \in \mathbb{N}^*),$$

then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

PROOF. Let $h(z)$ be defined by (9). Since of $f_i \in \mathcal{S}_n(p)$, using (4) in (13) and in view of (15) one may have

$$\left|\frac{zh''(z)}{h'(z)}\right| \quad \le \quad \sum_{i=1}^{k} \frac{1 + (1 + p|z|^n)M}{|\alpha|} < \frac{k(1 + (1 + p)M)}{|\alpha|} = \gamma_1 \quad (z \in U).$$

The proof can now be completed as in the proof of Theorem 1.

REMARK 2. By taking $n = 2$, Theorem 2 is reduced to Theorem E.

THEOREM 3. Let $\alpha, \beta \in \mathbb{C}$, $\Re\beta \ge \gamma_2$ and

$$\gamma_2 := \sum_{i=1}^{k} \frac{\beta_i}{|\alpha|} \quad (0 < \beta_i \le 1, \ i = 1, 2, ..., k, \ k \in \mathbb{N}^*). \tag{16}$$

If $f_i \in \mathcal{A}_n$ $(i = 1, 2, ..., \ k \in \mathbb{N}^*)$ defined by (7) satisfies the conditions

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < \beta_i \quad (0 < \beta_i \le 1, \ z \in U, \quad i = 1, 2, ..., k, \ k \in \mathbb{N}^*), \tag{17}$$

then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

PROOF. From (10), it follows that

$$\left|\frac{zh''(z)}{h'(z)}\right| = \left|\sum_{i=1}^{n} \frac{1}{\alpha}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right| \le \sum_{i=1}^{n} \frac{1}{|\alpha|}\left|\frac{zf_i'(z)}{f_i(z)} - 1\right|. \tag{18}$$

On making use of (17), (18) and in view of (16), the relation obtained is

$$\left|\frac{zh''(z)}{h'(z)}\right| \quad < \quad \sum_{i=1}^{k} \frac{\beta_i}{|\alpha|} = \gamma_2.$$

The remaining part of the proof is similar to the proof of Theorem 1.

By taking $\beta_i = 1$ $(i = 1, 2, ..., k, \ k \in \mathbb{N}^*)$ in Theorem 3, the following result is obtained.

EXAMPLE. Let $\alpha, \beta \in \mathbb{C}$, $\Re\beta \ge \frac{k}{|\alpha|}$. If $f_i \in \mathcal{A}_n$ defined by (7) satisfies

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1 \quad (z \in U, \quad i = 1, 2, ..., k),$$

then $F_{\alpha,\beta}(z) \in \mathcal{S}$.

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