Unique Periodic Solution Of ES-S Model*

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Abstract

In this paper, coincidence degree method is employed to prove existence of $T$-periodic solutions in $D$ for a non-autonomous system which is called ES-S model, where $D$ is a strictly positively invariant region of ES-S model. Furthermore, Floquet theory is provided to analyze uniqueness of a $T$-periodic solution of ES-S model.

1 Introduction

Schnakeberg model [7] is

$$\begin{align*}
  u_t(r,t) &= d_1 \Delta u(r,t) + a - u(r,t) + u^2(r,t)v(r,t), \quad r \in \Lambda, \\
  v_t(r,t) &= d_2 \Delta v(r,t) + b - u^2(r,t)v(r,t), \quad r \in \Lambda
\end{align*}$$

with boundary conditions

$$n(r) \cdot \nabla u(r,t) = n(r) \cdot \nabla v(r,t) = 0$$

for $r \in \partial \Lambda$, where $n(r)$ is the unit outward normal vector field along the boundary of $0\Lambda = [0,l] \times [0,l]$ ($l > 0$) and $a, b, d_1, d_2$ are positive constants. If we consider the case when reactants are well stirred, then the diffusion terms disappear. In this case, we get the simplified Schnakeberg (S-S) model [5, p. 156]

$$\begin{align*}
  \dot{u} &= a - u + u^2v \\
  \dot{v} &= b - u^2v.
\end{align*}$$

If in S-S model, we allow the coefficients $a$ and $b$ to be positive continuous $T$-periodic functions of $t$ with period $T > 0$, then the corresponding model is called ES-S model. Similarly, Schnakeberg model will be called ES model, if we replace constants $a$ and $b$ by positive continuous $T$-periodic functions.

Our goal is to concentrate on the study of periodic patterns of ES model. This is a job different from the study of pattern formation of Schnakeberg model by using Turing’s Instability [5, p. 380]. In order to study the pattern formation of ES model, we need to find a $T$-periodic solution (a source of the $T$-periodic pattern) for ES-S model in a certain patch.

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2 Main Results

\[ x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{and} \quad F(t, x(t)) = \begin{pmatrix} a - u + u^2v \\ b - u^2v \end{pmatrix}. \]

Then ES-S model is defined by

\[ \dot{x}(t) = F(t, x(t)) \]

with conditions

\[ 1.1 < a(t) < 1.6, \quad 0.04 < b(t) < 0.1. \]

**Lemma 1.** There exists a strictly positively invariant region

\[ D = \{(u, v) \in \mathbb{R}^2 : 1 \leq u \leq 2, \quad 0.01 \leq v \leq 0.1\} \]

for ES-S model given by (1) with conditions (2).

Linearize the system (1) with respect to its a \( T \)-periodic solution \( x(t) = (u(t), v(t))^T \in D \) for any \( t \in \mathbb{R} \) (if such a \( T \)-periodic solution exists), then we get

\[ \dot{W}(t) = A(t)W(t), \]

where

\[ A(t) = F_x'(t) = \begin{pmatrix} -1 + 2u(t)v(t) & u^2(t) \\ -2u(t)v(t) & -u^2(t) \end{pmatrix}, \]

\[ W(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \]

is a variation vector field along the trajectory \( x(t) \).

**Proposition 2.** Linear system (3) satisfies \( \text{tr}(A(t)) < 0 \) and \( \det(A(t)) > 0 \) for any \( t \in \mathbb{R} \).

The similar proofs of Lemma 1 and Proposition 2 are in [3]. Now, let us state the main result:

**Theorem 3.** For ES-S model with conditions (2), there exists only one \( T \)-periodic solution \( x_0(t) \) in \( D \).

3 Preliminaries

Let \( \mathcal{X} = \{x \in C([0, T]) \mid x(0) = x(T)\} \). Clearly \( \mathcal{X} \) is a Banach space with the supremum norm. Define \( Lx(t) = \dot{x}(t) \) with domain

\[ \text{Dom}(L) = \{x \in C^1([0, T]) \mid x(0) = x(T)\}. \]

It is easy to verify that \( \text{Dom}(L) \) is contained in \( \mathcal{X} \), the range of \( L \) is \( \text{Im}(L) = \{z(t) \in \mathcal{X} \mid \int_0^T z(t)dt = 0\} \) and \( L \) is a Fredholm mapping of index 0. Let

\[ \Theta = \{x \in \text{Dom}(L) \mid x(t) \in D, \forall t \in [0, T]\}. \]
Define $F_1 : \Theta \rightarrow \mathcal{X}$ by $F_1(x) = F(., x(\cdot))$ and $H_1(x(t)) = F_1(x(t)) - Lx(t)$.

Now, construct a homotopy family $H_\lambda : (\text{Dom}(L) \cap \Theta) \times [0, 1] \rightarrow \mathcal{X}$

to be of the form

$$H_\lambda(x(t)) = \mathcal{F}_\lambda(x(t)) - Lx(t),$$

where $\mathcal{F}_\lambda : \Theta \times [0, 1] \rightarrow \mathcal{X}$

with

$$\mathcal{F}_\lambda(x(t)) = \begin{pmatrix} \tilde{a}(t) - u(t) + u^2(t)v(t) \\ \tilde{b}(t) - u^2(t)v(t) \end{pmatrix}.$$  

Here $\tilde{a}(t) = (1 - \lambda)1.4 + \lambda a(t)$, and $\tilde{b}(t) = 0.05(1 - \lambda) + \lambda b(t)$ with $\lambda \in [0, 1]$. It is easy to verify that $\mathcal{F}_\lambda : \Theta \times [0, 1] \rightarrow \mathcal{X}$ is $L$-compact. For more details of degree theory, see [4, Ch. I–IV].

**LEMMA 4.** Given $\lambda \in [0, 1]$, if $x(t) \in \Theta$ is a $T$-periodic solution of the system

$$\dot{x}(t) = \mathcal{F}_\lambda(x(t)),$$

then $\partial \mathcal{D}$ is an a priori bound of $x(t)$.

**PROOF.** Clearly, $\tilde{a}$ and $\tilde{b}$ satisfy conditions (2). System (7) is an ES-S model. By Lemma 1, $\mathcal{D}$ is still a strictly positively invariant region of system (7). None of $T$-periodic solutions of (7) in $\Theta$ can touch the boundary of $\mathcal{D}$.

**COROLLARY 5.** $0 \not\in H_\lambda((\text{Dom}(L) \cap \partial \mathcal{D}) \times [0, 1])$.

**LEMMA 6.** $D_L(H_0(x(t), \Theta) = D_B(H_0(x(t), \mathcal{D}) = 1$, where $D_L$ denote Leray-Schauder degree and $D_B$ denote Brouwer degree.

For a similar proof, see [2].

**LEMMA 7.** For system (3) with conditions (2), zero is the only $T$-periodic solution.

**PROOF.** Suppose (3) has a non-trivial $T$-periodic solution called $W_1(t)$. By Proposition 2 and Floquet theory [1, p. 93-105], its orbit $\Gamma$ is an orbitally asymptotically stable. For $s \in \mathbb{R}$, $sW_1(t)$ is also a $T$-periodic solution of (3). Then orbit of $sW_1(t)$ can not be attracted to $\Gamma$ for any $s \in \mathbb{R}$. This leads a contradiction to the orbital asymptotic stability of $\Gamma$.

**REMARK 8.** For the linear system (3), if $\text{tr}(A(t))$ does not change sign in some simply connected region $E \subset \mathbb{R}^2$, then (3) has no non-trivial periodic solution in $E$; since the system (3) is a linearization of an non-autonomous system, Bendixson’s Criteria [6, p. 264] cannot be used to prove Lemma 7.

## 4 Proof of Theorem 3

**PROOF.** *(Existence)* Combine Lemma 4, Corollary 5 and Lemma 6, by a general existence theorem of the Leray-Schauder type, we get

$$D_L(H_1(x(t)), \Theta) = D_L(H_0(x(t)), \Theta) = D_B(F_0(x(t)), \mathcal{D}) = 1,$$
which implies that there at least exists one \( T \)-periodic solution \( x_0(t) = (u_0(t), v_0(t))^T \) of ES-S model in \( D \). If \( a \) and \( b \) are constants, it is easy to show that there is only one trivial \( T \)-periodic solution \( x_0 \in \text{int}(D) \), otherwise, we can easily verify that \( x_0(t) \) is a nontrivial \( T \)-periodic solution of ES-S model in \( D \) by substituting \( x_0(t) \) into ES-S model.

(Unique\( \text{n}\)ness) Define \( C_T = \{ x(t) \in \Theta | x(t) \text{ satisfies (1) with conditions (2)} \} \). Since \( x_0(t) \in C_T \), \( C_T \) is not an empty set. If \( a \) and \( b \) are constants, there is only one constant solution in \( C_T \).

If one of \( a(t) \) and \( b(t) \) is a non-trivial \( T \)-periodic function, then \( x_0(t) \in C_T \) is a non-trivial \( T \)-periodic solution. Assume \( C_T \) is not a singleton; we pick

\[
x_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix}
\]
in \( C_T \) and substitute them into (1) to get

\[
x_i(t) = F(t, x_i(t)), \quad i = 1, 2.
\]

Define \( z(t) = x_1(t) - x_2(t) \). By the mean value theorem, we get

\[
\dot{z}(t) = z(t) \int_0^1 F_x'(t, x_2(t) + \theta(x_1(t) - x_2(t)))d\theta
\]
and

\[
\int_0^1 F_x'(t, x_2(t) + \theta(x_1(t) - x_2(t)))d\theta = \begin{pmatrix} -1 + 2n(t) & m(t) \\ -2n(t) & -m(t) \end{pmatrix},
\]
where

\[
m(t) = \int_0^1 [v_2(t) + \theta(v_1(t) - v_2(t))]^2d\theta,
\]
\[
n(t) = \int_0^1 [u_2(t) + \theta(u_1(t) - u_2(t))]^2[v_2(t) + \theta(v_1(t) - v_2(t))]d\theta.
\]

Since

\[
n(t) = \frac{1}{2}(v_2(t)u_1(t) + v_2(t)v_1(t)) + \frac{1}{3}(u_1(t) - u_2(t))(v_1(t) - v_2(t))
\]
and

\[
m(t) = \frac{1}{3}(v_1(t) - v_2(t))^2 + v_2(t)v_1(t),
\]

\[
tr \left( \int_0^1 F_x'(t, x_2(t) + \theta(x_1(t) - x_2(t)))d\theta \right) = -1 - m(t) + 2n(t)
\]
\[
= -1 - \frac{1}{3}v_1(t)^2 - \frac{1}{3}v_2(t)^2 - \frac{1}{3}v_1(t)v_2(t)
\]
\[
+ \frac{1}{3}v_2(t)u_1(t) + \frac{1}{3}v_1(t)u_2(t) + \frac{2}{3}v_1(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t).
\]

Notice the following facts:

\[
\frac{1}{3}v_2(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t) \leq \frac{v_2(t)}{3}(u_1(t) + 2u_2(t)) \leq 2v_2(t) \leq 0.2,
\]
\[
\frac{1}{3} v_1(t)u_2(t) + \frac{2}{3} v_1(t)u_1(t) \leq \frac{v_1(t)}{3}(u_2(t) + 2u_1(t)) \leq 2v_1(t) \leq 0.2.
\]

It follows that
\[
tr \left( \int_0^1 F_x'[t, x_2(t) + \theta(x_1(t) - x_2(t))]d\theta \right) < 0. \tag{10}
\]

This implies that the zero solution is the only one \(T\)-periodic solution for (9) by Remark 8. Hence, \(x_1(t) = x_2(t)\). \(C_T\) is a singleton.

## 5 Future Works

In this paper, we proved the existence and uniqueness of the periodic solution \(x_0(t)\) of ES-S model. This establishes a foundation for further studying patterns of ES model. Of course, the problem mentioned here is still open, the investigation of this question is currently underway.

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## References


