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Unique Periodic Solution Of ES-S Model*

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Abstract

In this paper, coincidence degree method is employed to prove existence of T-periodic solutions in $\mathcal{D}$ for a non-autonomous system which is called ES-S model, where $\mathcal{D}$ is a strictly positively invariant region of ES-S model. Furthermore, Floquet theory is provided to analyze uniqueness of a T-periodic solution of ES-S model.

1 Introduction

Schnakeberg model [7] is

$$\begin{cases} u_t(r,t) = d_1\Delta u(r,t) + a - u(r,t) + u^2(r,t)v(r,t), & r \in \Lambda, \\ v_t(r,t) = d_2\Delta v(r,t) + b - u^2(r,t)v(r,t), & r \in \Lambda \end{cases}$$

with boundary conditions

$$n(r) \cdot \nabla u(r,t) = n(r) \cdot \nabla v(r,t) = 0$$

for $r \in \partial\Lambda$, where $n(r)$ is the unit outward normal vector field along the boundary of $0\Lambda = [0,l] \times [0,l]$ $(l > 0)$ and a, b, d_1, d_2 are positive constants. If we consider the case when reactants are well stirred, then the diffusion terms disappear. In this case, we get the simplified Schnakeberg (S-S) model [5, p. 156]

$$\begin{cases} \dot{u} = a - u + u^2 v \\ \dot{v} = b - u^2 v. \end{cases}$$

If in S-S model, we allow the coefficients a and b to be positive continuous T-periodic functions of t with period $T > 0$, then the corresponding model is called ES-S model. Similarly, Schnakeberg model will be called ES model, if we replace constants a and b by positive continuous T-periodic functions.

Our goal is to concentrate on the study of periodic patterns of ES model. This is a job different from the study of pattern formation of Schnakeberg model by using Turing's Instability [5, p.380]. In order to study the pattern formation of ES model, we need to find a T-periodic solution (a source of the T-periodic pattern) for ES-S model in a certain patch.

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2 Main Results

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{and} \quad F(t, x(t)) = \begin{pmatrix} a - u + u^2 v \\ b - u^2 v \end{pmatrix}.$$

Then ES-S model is defined by

$$\dot{x}(t) = F(t, x(t)) \tag{1}$$

with conditions

$$1.1 < a(t) < 1.6, \quad 0.04 < b(t) < 0.1. \tag{2}$$

LEMMA 1. There exists a strictly positively invariant region

$$\mathcal{D} = \{(u, v) \in \mathbb{R}^2 \,:\, 1 \le u \le 2,\ 0.01 \le v \le 0.1\}$$

for ES-S model given by (1) with conditions (2).

Linearize the system (1) with respect to its a T-periodic solution $x(t) = (u(t), v(t))^T \in \mathcal{D}$ for any $t \in \mathbb{R}$ (if such a T-periodic solution exists), then we get

$$\dot{W}(t) = A(t)W(t), \tag{3}$$

where

$$A(t) = F'_{x(t)} = \begin{pmatrix} -1 + 2u(t)v(t) & u^2(t) \\ -2u(t)v(t) & -u^2(t) \end{pmatrix}.$$

$$W(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

is a variation vector field along the trajectory $x(t)$.

PROPOSITION 2. Linear system (3) satisfies $tr(A(t)) < 0$ and $det(A(t)) > 0$ for any $t \in \mathbb{R}$.

The similar proofs of Lemma 1 and Proposition 2 are in [3]. Now, let us state the main result:

THEOREM 3. For ES-S model with conditions (2), there exists only one T-periodic solution $x_0(t)$ in $\mathcal{D}$.

3 Preliminaries

Let $\mathcal{X} = \{x \in C([0, T]) \,|\, x(0) = x(T)\}$. Clearly $\mathcal{X}$ is a Banach space with the supremum norm. Define $Lx(t) = \dot{x}(t)$ with domain

$$Dom(L) = \{x \in C^1([0, T]) \,|\, x(0) = x(T)\}.$$

It is easy to verify that $Dom(L)$ is contained in $\mathcal{X}$, the range of L is $Im(L) = \{z(t) \in \mathcal{X} \,|\, \int_0^T z(t)dt = 0\}$ and L is a Fredholm mapping of index 0. Let

$$\Theta = \{x \in Dom(L) \,|\, x(t) \in \mathcal{D}, \quad \forall t \in [0, T]\}. \tag{4}$$

Define $\mathcal{F}_1 : \Theta \to \mathcal{X}$ by $\mathcal{F}_1(x) = F(\cdot, x(\cdot))$ and $H_1(x(t)) = \mathcal{F}_1(x(t)) - Lx(t)$.

Now, construct a homotopy family

$$H_\lambda : (Dom(L) \cap \Theta) \times [0,1] \to \mathcal{X}$$

to be of the form

$$H_\lambda(x(t)) = \mathcal{F}_\lambda(x(t)) - Lx(t), \tag{5}$$

where $\mathcal{F}_\lambda : \Theta \times [0,1] \to \mathcal{X}$ with

$$\mathcal{F}_\lambda(x(t)) = \begin{pmatrix} \tilde{a}(t) - u(t) + u^2(t)v(t) \\ \tilde{b}(t) - u^2(t)v(t) \end{pmatrix}. \tag{6}$$

Here $\tilde{a}(t) = (1-\lambda)1.4 + \lambda a(t)$, and $\tilde{b}(t) = 0.05(1-\lambda) + \lambda b(t)$ with $\lambda \in [0,1]$. It is easy to verify that $\mathcal{F}_\lambda : \overline{\Theta} \times [0,1] \to \mathcal{X}$ is L-compact. For more details of degree theory, see [4, Ch. I–IV].

LEMMA 4. Given $\lambda \in [0,1]$, if $x(t) \in \Theta$ is a T-periodic solution of the system

$$\dot{x}(t) = \mathcal{F}_\lambda(x(t)), \tag{7}$$

then $\partial\mathcal{D}$ is an a priori bound of $x(t)$.

PROOF. Clearly, $\tilde{a}$ and $\tilde{b}$ satisfy conditions (2). System (7) is an ES-S model. By Lemma 1, $\mathcal{D}$ is still a strictly positively invariant region of system (7). None of T-periodic solutions of (7) in Θ can touch the boundary of $\mathcal{D}$.

COROLLARY 5. $0 \not\in H_\lambda((Dom(L) \cap \partial\mathcal{D}) \times [0,1])$.

LEMMA 6. $D_L(H_0(x(t), \Theta) = D_B(H_0(x(t), \mathcal{D}) = 1$, where D_L denote Leray-Schauder degree and D_B denote Brouwer degree.

For a similar proof, see [2].

LEMMA 7. For system (3) with conditions (2), zero is the only T-periodic solution.

PROOF. Suppose (3) has a non-trivial T-periodic solution called $W_1(t)$. By Proposition 2 and Floquet theory [1, p. 93-105], its orbit Γ is an orbitally asymptotically stable. For $s \in \mathbb{R}$, $sW_1(t)$ is also a T-periodic solution of (3). Then orbit of $sW_1(t)$ can not be attracted to Γ for any $s \in \mathbb{R}$. This leads a contradiction to the orbital asymptotic stability of Γ.

REMARK 8. For the linear system (3), if $tr(A(t))$ does not change sign in some simply connected region $E \subset \mathbb{R}^2$, then (3) has no non-trivial periodic solution in E; since the system (3) is a linearization of an non-autonomous system, Bendixson's Criteria [6, p. 264] cannot be used to prove Lemma 7.

4 Proof of Theorem 3

PROOF. (**Existence**) Combine Lemma 4, Corollary 5 and Lemma 6, by a general existence theorem of the Leray-Schauder type, we get

$$D_L(H_1(x(t)), \Theta) = D_L(H_0(x(t)), \Theta) = D_B(\mathcal{F}_0(x(t)), \mathcal{D}) = 1,$$

which implies that there at least exists one T-periodic solution $x_0(t) = (u_0(t), v_0(t))^T$ of ES-S model in $\mathcal{D}$. If a and b are constants, it is easy to show that there is only one trivial T-periodic solution $x_0 \in int(\mathcal{D})$, otherwise, we can easily verify that $x_0(t)$ is a nontrivial T-periodic solution of ES-S model in $\mathcal{D}$ by substituting $x_0(t)$ into ES-S model.

(**Uniqueness**) Define $C_T = \{x(t) \in \Theta | x(t)$ satisfies (1) with conditions (2)$\}$. Since $x_0(t) \in C_T$, C_T is not an empty set. If a and b are constants, there is only one constant solution in C_T.

If one of $a(t)$ and $b(t)$ is a non-trivial T-periodic function, then $x_0(t) \in C_T$ is a non-trivial T-periodic solution. Assume C_T is not a singleton; we pick

$$x_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \;\; x_2(t) = \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix}$$

in C_T and substitute them into (1) to get

$$\dot{x_i}(t) = F(t, x_i(t)), \quad i = 1, 2. \tag{8}$$

Define $z(t) = x_1(t) - x_2(t)$. By the mean value theorem, we get

$$\dot{z}(t) = z(t) \int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta \tag{9}$$

and

$$\int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta = \begin{pmatrix} -1 + 2n(t) & m(t) \\ -2n(t) & -m(t) \end{pmatrix},$$

where

$$m(t) = \int_0^1 [v_2(t) + \theta(v_1(t) - v_2(t))]^2 d\theta,$$

$$n(t) = \int_0^1 [u_2(t) + \theta(u_1(t) - u_2(t))][v_2(t) + \theta(v_1(t) - v_2(t))] d\theta.$$

Since

$$n(t) = \frac{1}{2}(v_2(t)u_1(t) + u_2(t)v_1(t)) + \frac{1}{3}(u_1(t) - u_2(t))(v_1(t) - v_2(t))$$

and

$$m(t) = \frac{1}{3}(v_1(t) - v_2(t))^2 + v_2(t)v_1(t),$$

$$tr\left(\int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))] d\theta\right) = -1 - m(t) + 2n(t)$$

$$= -1 - \frac{1}{3}v_1(t)^2 - \frac{1}{3}v_2(t)^2 - \frac{1}{3}v_1(t)v_2(t)$$

$$+ \frac{1}{3}v_2(t)u_1(t) + \frac{1}{3}v_1(t)u_2(t) + \frac{2}{3}v_1(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t).$$

Notice the following facts:

$$\frac{1}{3}v_2(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t) \le \frac{v_2(t)}{3}(u_1(t) + 2u_2(t)) \le 2v_2(t) \le 0.2,$$

$$\frac{1}{3}v_1(t)u_2(t) + \frac{2}{3}v_1(t)u_1(t) \leq \frac{v_1(t)}{3}(u_2(t) + 2u_1(t)) \leq 2v_1(t) \leq 0.2.$$

It follows that

$$tr\left(\int_0^1 F_x^{'}[t, x_2(t) + \theta(x_1(t) - x_2(t))]d\theta\right) < 0. \tag{10}$$

This implies that the zero solution is the only one T-periodic solution for (9) by Remark 8. Hence, $x_1(t) = x_2(t)$. C_T is a singleton.

5 Future Works

In this paper, we proved the existence and uniqueness of the periodic solution $x_0(t)$ of ES-S model. This establishes a foundation for further studying patterns of ES model. Of course, the problem mentioned here is still open, the investigation of this question is currently underway.

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