Applied Mathematics E-Notes, 8(2008), 17-24 © ISSN 1607-2510
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Existence And Uniqueness Of The Solution of Laplace's Equation From A Model Of Magnetic Recording*

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Received 1 November 2006

Abstract

A variational formulation is given for the solution of Laplace's equation within an infinitely deep rectagular gap due to a source charge existing in a half space outside the gap. The problem arises in the modeling of the magnetic recording process. Ad hoc Fourier based solutions and finite element formulations are used by engineers to approximate solutions of the problem. However, despite the relatively simple equation and geometry, the appropriate variational formulation of the problem has never been given. The variational form is essential to confirm the veracity of numerical schemes applied to the problems as well as to provide the basis for numerical analysis of the schemes. The formulation will require two Dirichlet to Neumann maps. One map characterizes the interface from the half space to the gap. The second map truncates the solution domain within the gap itself. The maps will allow the problem to be formulated on a finite rectangular domain while properly representing the influence of the source charge which is external to the solution domain. The paper will offer the proper formulation of these boundary conditions and provide a proof of the existence and uniqueness of the solution to the variational problem.

1 Introduction

In computer hard drives, information is stored on a disk of magnetic material. The information is stored in concentric circles called tracks and each track is divided into sectors. In each sector a magnetization is printed with one of two distinct orientations. If the orientation changes between sectors in a track, this represents the binary digit 1. If there is no change in orientation this is the binary digit 0. The magnetization of the media on the disk creates a magnetic charge distribution ρ which induces a magnetic potential ϕ in the surrounding region. Reading the information on the disk requires detection the potential with the use of a magneto-resistive material. The magneto-resistive material is placed between two magnetic shields above the disk media. The

*Mathematics Subject Classifications: 35J05,35J20,78A30.

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shields have infinite permeability and serve to focus the potential produced by the media magnetization onto the sensor [7].

The paper will give a description of a two dimensional model of the situation described above. Some simplifying assumptions will be made regarding the geometry of the problem. The solution of the problem is a magnetic potential ϕ which satisfies Laplace's equation $\triangle\phi = 0$. The unique aspect of the model is the necessity of two special boundary conditions which allow the problem to be limited to a finite domain. The paper will provide the proper formulation of the boundary conditions based on the Fourier theory solution of the problem. The Fourier based approach for the problem is given in [5] and a similar solution is given in [3]. The same basic approach is used for the Helmholtz equation in [1]. None of the sources provide a mathematical proof of existence and uniqueness of the solution, but the ideas can be used to develop the proper mathematical formulation of the problem.

Once the boundary conditions are established, a variational formulation in the finite domain will be provided. The proof of existence and uniqueness of the solution of the variational formulation is also given. The proof involves methods from functional analysis. The existence and uniqueness proof is not only of pure mathematical interest. It is also important since it will provide the foundation for the analysis of numerical methods which are applied to compute approximate solutions of the model such as finite elements and finite Fourier methods.

2 Problem Description

The analysis of the problem will be done in two dimensions. Therefore the assumption is made that there is minimal variation in the z-direction. In the xy-plane, the upper half plane ($y > 0$) consists of a magnetic shield with an empty gap from $x = 0$ to $x = G$. The gap is where a sensor would be placed to detect variations in the potential produced by a charge outside the gap. We will refer to the region in the upper half plane inside the gap as region I. (See figure 1) In the lower half space ($y < 0$) there will be no material but only a magnetic charge distribution. The magnetic charge ρ represents information on a magnetic media below the gap. We will refer to the entire lower half plane as region II. The assumption is made that the variations of the magnetic charge below the gap are very small with respect to the size of the magnetic shields. The size of the shields justifies the assumption that the gap is infinitely deep in the y-direction as well as infinitely wide in the x-direction (see figure 1). In actual magnetic recording there is a very thin sensor placed in the gap between the magnetic shields which is used to detect the potential produced from the charge outside the gap and hence read the stored information. As a final assumption, the material of the sensor is not considered in the gap region.

Note that the for any region not containing a magnetic charge, the magnetic potential will satisfy Laplace's equation $\triangle\phi = 0$. Also, note that the shields do not support a magnetic potential. Therefore, any boundary with the shields will have a Dirichlet boundary condition $\phi = 0$. Denote the potential in region I as ϕ_I and the potential below the shields in region II as ϕ_{II}. A special boundary condition will be provided at the interface $y = 0$. The boundary condition will ensure continuity of the potential and its normal derivative across the interface. In addition, an artificial boundary will be placed in the gap at some positive value $y = b$ inside the gap in region I. The artificial boundary will create a finite rectangular domain on which the analysis of the problem will be done. The region within the shield gap, above the interface and below the artificial boundary will be denoted $S = [0, G] \times [0, b]$. The boundary of the region S is denoted Γ. In particular, the portion of the boundary at $y = b$ is denoted Γ_1 while the portion of the boundary at $y = 0$ is denoted Γ_2. The goal of the paper is to describe the proper formulation of the differential equation and boundary conditions which ϕ_I satisfy in S and give a proof that the formulation has a unique solution.

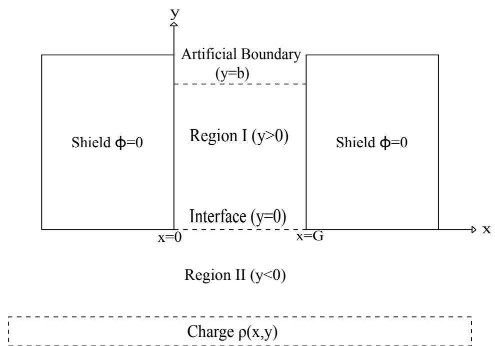

Figure 1. Diagram of solution S domain for the variational problem

3 Boundary Conditions

The most interesting aspect of the method is what happens at Γ_2 which is the transition from the empty lower half space below the shields to the gap between the shields. In region I, the solution will have the form

$$\phi_I = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{G}\right) e^{-\left(\frac{n\pi}{G}\right)y} \tag{1}$$

Since the support of the potential in the x-direction, is nonzero only in the interval from 0 to G, a Fourier sine series can be used [6]. On the other hand in region II outside the gap and above the charge distribution, the solution has the form

$$\phi_{II} = \int_{-\infty}^{+\infty} \left(B(k_x)e^{-\kappa y} + C(k_x)e^{\kappa y}\right) e^{2\pi i k_x x}\, dk_x \tag{2}$$

(Where $\kappa = |2\pi k_x|$) Since the support in the x direction is now infinite, a Fourier transform is used instead of a Fourier series to represent the solution [6]. The interesting part of the problem comes from coupling the two forms of the solution together at the gap interface Γ_2.

We can compute the potential without the presence of shields (this will be denoted by ϕ_0) exactly using a standard Green's function [2]. The potential ϕ_o in terms of the

Fourier components is

$$\phi_0 = \int_{-\infty}^{+\infty} B(k_x) e^{-\kappa y}\, dk_x. \tag{3}$$

Note that this form is due to the fact that the quantity ϕ_o must decay as $y > 0$ increases to be physically reasonable.

Also, note that the potential reflected from the bottom of the shields into region II (ϕ_r) is of the form

$$\phi_r = \int_{-\infty}^{+\infty} C(k_x) e^{\kappa y}\, dk_x \tag{4}$$

Note that ϕ_r must decay as $y < 0$ decreases in order to be physically reasonable. Also, note that C is an unknown quantity unlike B which can be computed directly. The total solution in region II is simply the superposition of the potentials ϕ_0 and ϕ_r. Examine the interface between the gap in region I and the lower half-plane of region II. The two forms of the solution will satisfy some standard continuity conditions.

First, the potential is continuous at the gap interface which implies the following equation

$$\left.\phi_I\right|_{y=0} = \left.\phi_{II}\right|_{y=0}. \tag{5}$$

Second, the normal derivative $\left(\frac{\partial}{\partial y}\right)$ is continuous across the gap interface. Hence, we have the equation

$$\left.\frac{\partial \phi_I}{\partial y}\right|_{y=0} = \left.\frac{\partial \phi_{II}}{\partial y}\right|_{y=0} \tag{6}$$

Using the forms of the solution in the respective regions (1) and (2) the boundary condition at the interface between region I and region II can be found.

The first continuity equation gives the following:

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{G}\right) = \int_{-\infty}^{+\infty} (B + C)\, e^{2\pi i k_x x}\, dk_x \tag{7}$$

or

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{G}\right) = \mathcal{F}_x^{-1}(B + C) \tag{8}$$

Taking the Fourier Transform in the x-direction of both sides gives

$$\sum_{n=1}^{\infty} A_n \mathcal{F}_x\left(\sin\left(\frac{n\pi x}{G}\right)\right) = (B + C) \tag{9}$$

or

$$\mathcal{F}_x(\phi_I) = B + C \tag{10}$$

Note that both the A_n's and C are unknown. Hence, it will be necessary to eliminate C and leave the expression in terms of the known quantity B.

The second continuity condition gives the following:

$$\sum_{n=1}^{\infty}\left(\frac{-n\pi}{G}\right)A_n\sin\left(\frac{n\pi x}{G}\right)=\int\limits_{-\infty}^{+\infty}\left(-\kappa B+\kappa C\right)e^{2\pi i k_x x}\,dk_x \tag{11}$$

or

$$\frac{\partial\phi_I}{\partial y}=\sum_{n=1}^{\infty}\left(\frac{-n\pi}{G}\right)A_n\sin\left(\frac{n\pi x}{G}\right)=\mathcal{F}_x^{-1}(-\kappa B+\kappa C) \tag{12}$$

Equation (11) is the boundary condition which will be enforced on Γ_2. On Γ_1 the solution ϕ_I will be of the form

$$\phi_I=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{G}\right)e^{-\left(\frac{n\pi}{G}\right)y} \tag{13}$$

Therefore, on the artificial boundary Γ_1

$$\frac{\partial\phi_I}{\partial y}=\sum_{n=1}^{\infty}\left(\frac{-n\pi}{G}\right)A_n\sin\left(\frac{n\pi x}{G}\right)e^{-\left(\frac{n\pi}{G}\right)b} \tag{14}$$

Both the expressions for $\frac{\partial\phi_I}{\partial y}$ will serve as the basis for the boundary conditions in the variational formulation of the problem given below.

4 Variational Formulation

The variational approach to the problem not only provides a basis for mathematical proofs of existence and uniqueness, but also provides the foundation for robust numerical methods including the finite element method. We look for a unique weak solution u of the Laplace equation $\triangle u=0$ in S using the boundary conditions described above in an appropriate space of functions.

Look for a solution in the rectangular region S from $x=0$ to $x=G$ and $y=0$ to $y=b$. In order to enforce the Dirichlet conditions $u\left(0,y\right)=u\left(G,y\right)=0$, the problem is restricted to the space of functions

$$\widetilde{H}_0^1\left(S\right)=\left\{w\in H^1(S)\mid w(0,y)=w(G,y)=0\right\} \tag{15}$$

The weak form of the problem is constructed as follows:

1. Multiply both sides of the Laplace equation $\triangle u=0$ equation by a function w in $\widetilde{H}_0^1\left(S\right)$ and integrate over S

$$\int_S\triangle uw\,dS=0 \tag{16}$$

2. Apply integration by parts to arrive at

$$-\int_S\nabla u\cdot\nabla w\,dS+\int_\Gamma\frac{\partial u}{\partial n}w\,d\Gamma=0 \tag{17}$$

Before moving on to the variational solution, examine the boundary term of the weak form of the equation. In particular, this term is where we insert the conditions described in section 3.

Note that $w(0,y) = w(G,y) = 0$, but boundary conditions on $\Gamma_1(y = b)$ and $\Gamma_2(y = 0)$ must also be specified. Hence, a Dirichlet to Neumann map $T_1(u)$ is defined on Γ_1 using equation (14),

$$\frac{\partial u}{\partial n} = T_1(u) = \sum_{n=1}^{\infty} \left(\frac{-n\pi}{G}\right) A_n \sin\left(\frac{n\pi x}{G}\right) \mathrm{e}^{-(\frac{n\pi}{G})b} \tag{18}$$

On the boundary Γ_2, refer back to equation (11):

$$\frac{\partial u}{\partial y} = \mathcal{F}_x^{-1}(-\kappa B + \kappa C) \tag{19}$$

Note that (10) is solved for C and then substituted into (12) to arrive at

$$\frac{\partial u}{\partial y} = \mathcal{F}_x^{-1}(-2\kappa B + \kappa \mathcal{F}_x(u)) \tag{20}$$

or

$$\frac{\partial u}{\partial y} = \mathcal{F}_x^{-1}(\kappa \mathcal{F}_x(u)) + g \tag{21}$$

where $g = -2\mathcal{F}_x^{-1}(\kappa B)$.

Hence, a Dirichlet to Neumann map $T_2(u)$ is defined on Γ_2 as the following:

$$\frac{\partial u}{\partial n} = T_2(u) - g = -\mathcal{F}_x^{-1}(\kappa \mathcal{F}_x(u)) - g \tag{22}$$

Now, put the maps $T_1(u)$ and $T_2(u)$ into the weak form of the equation (17)

$$-\int_S \nabla u \cdot \nabla w \, dS + \int_{\Gamma_1} T_1(u) w \, d\Gamma + \int_{\Gamma_2} T_2(u) w \, d\Gamma = \int_{\Gamma_1} g w \, d\Gamma \tag{23}$$

The weak form of the equation defines a bilinear form

$$a(u,w) = -\int_S \nabla u \cdot \nabla w \, dS + \int_{\Gamma_1} T_1(u) w \, d\Gamma + \int_{\Gamma_2} T_2(u) w \, d\Gamma \tag{24}$$

and a bounded linear functional

$$\langle g, w\rangle = \int_{\Gamma_1} g w \, d\Gamma \tag{25}$$

The solution of the variational form of the problem is a function $u \in \widetilde{H}_0^1(S)$ such that

$$a(u,v) = \langle g, v\rangle \tag{26}$$

for all $v \in \widetilde{H}_0^1(S)$.

5 Existence and Uniqueness

The weak or variational formulation is used for a proof of existence and uniqueness of the problem at hand.

THEOREM. The variational problem (26) has a unique solution in $\widetilde{H}_0^1$.

PROOF. First, it must be shown that $a(u,v)$ is a continuous bilinear form on $\widetilde{H}_0^1(S)$. That is to say

$$a(u,v) \leq C\|u\|_{\widetilde{H}_0^1(S)}\|v\|_{\widetilde{H}_0^1(S)} \tag{27}$$

Next, establish coercivity of the bilinear form which means

$$a(u,u) \geq C\|u\|^2_{\widetilde{H}_0^1(S)} \tag{28}$$

Using the previous two facts, apply the Lax-Milgram Lemma to establish that there exists a unique solution to the differential equation in $\widetilde{H}_0^1(S)$ [4]. (From this point on $\|\cdot\| = \|\cdot\|_{\widetilde{H}_0^1(S)}$.)

First, establish the continuity of $a(u,v)$. Using the Cauchy-Schwartz inequality

$$\int_S \nabla\phi \cdot \nabla v \, dS \leq C\|u\|\|v\| \tag{29}$$

On the boundary Γ_1, let

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{G}\right) \mathrm{e}^{-\left(\frac{n\pi}{G}\right)b}$$

and

$$v = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{G}\right) \mathrm{e}^{-\left(\frac{n\pi}{G}\right)b}.$$

By orthogonality, the Cauchy-Schwartz inequality and the Trace Theorem

$$\int_{\Gamma_1} T_1(u)v \, d\Gamma = \sum_{n=1}^{\infty} \frac{n\pi}{G} e^{-2\left(\frac{n\pi}{G}\right)b} A_n B_n \leq C\|u\|\|v\|. \tag{30}$$

Now, by Parseval's Theorem, Cauchy-Schwartz inequality and the Trace Theorem

$$\int_{\Gamma_2} T_2(u)v \, d\Gamma = \int_{-\infty}^{+\infty} \kappa \mathcal{F}_x(u)\mathcal{F}_x^{-1}(v) \, dk_x \leq C\|u\|\|v\| \tag{31}$$

Thus, we have continuity

$$a(u,v) \leq C\|u\|\|v\| \tag{32}$$

Now, prove the coercivity of the bilinear form $a(u,v)$:

$$-a(u,u) = \int_S \nabla u^2 \, dS + \sum_{n=1}^{\infty} \frac{n\pi}{G} {A_n}^2 + \int_{-\infty}^{+\infty} \kappa \mathcal{F}_x(u)^2 \, dk_x \tag{33}$$

Poincaré's inequality indicates that

$$\int_S \nabla u^2 \, dS \geq C\|u\|^2 \tag{34}$$

Since the other two terms are clearly positive, we have the result

$$a(u,u) \geq C\|u\|^2 \tag{35}$$

Therefore, the conditions of Lax-Milgram Lemma are satisfied, and the existence of a unique solution $u \in \widetilde{H}_0^1(S)$ is guaranteed.

6 Conclusion

The simplified model of a magnetic recording process leads to a formulation which requires the solution of Laplace's equation in an infinitely deep gap. The solution is a magnetic potential which is induced from a charge existing entirely outside the gap. The formulation requires two special boundary conditions to reduce the problem to a finite solution domain. The paper has shown the proper formulation of the boundary conditions as well as a proof of the existence and uniqueness of the solution using a variational formulation.

References

[1] H. J. Eom, T. J. Park and K. Yoshitomi, An Analytic Solution for the Transverse-Magnetic Scattering From a Rectangular Channel in a Conducting Plane, J. Appl. Physics, 73(7)(1993), 3571-3573

[2] R. F. Harrington, Time-Harmonic Electromagnetic Fields, Wiley-IEEE Press, New York, 2001

[3] D. E. Heim, On the Track Profile in Magnetoresistive Heads, IEEE Trans. on Magnetics, 30(1994), 1453-1464.

[4] B. D. Reddy, Introductory Functional Analysis, Springer-Verlag, New York, 1998

[5] Y. Suzuki and C. Ishikawa, A New Method of Calculating the Medium Field and The Demagnetizing Field for MR Heads, IEEE Trans. on Magnetics, 34(1998), 1513-1515.

[6] G. P. Tolstov, Fourier Series, Dover Publications, INC., New York, 1962.

[7] S. X. Wang and A. M. Taratorin, Magnetic Information Storage Technology, Academic Press, San Diego, 1999