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On A System Of Equations Related To Bicentric Polygons*

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Abstract

We deal with bicentric n-gons where instead of incircle there is excircle. We also consider system of equations involving the different quantities associated with the n-gons, the circumcircles and the excirles.

1 Introduction

A bicentric polygon is a circumscribed polygon which also has an inscribed circle (a circle that is tangent to each side of the polygon). In [3], the following theorem is announced.

THEOREM A ([3, Theorem 1]). Let $A_1 \ldots A_n$ be any given bicentric n–gon. Let

$$\begin{aligned} R_0 = &\ \text{radius of circumcircle of } A_1 \ldots A_n, \\ r_0 = &\ \text{radius of incircle of } A_1 \ldots A_n, \text{ and} \\ d_0 = &\ \text{distance between centers of circumcircle and incircle.} \end{aligned}$$

Then there are lengths R_2, d_2, r_2 such that

$$R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2, \tag{1}$$

$$R_2 d_2 = R_0 d_0, \tag{2}$$

$$R_2^2 - d_2^2 = 2R_0 r_2. \tag{3}$$

It is not difficult to see that the positive solutions $R_{2\ell}, d_{2\ell}, r_{2\ell}$, $\ell = 1, 2$ in R_2, d_2, r_2 of the above system of equations satisfy

$$\begin{aligned} R_{21}^2 &= R_0 \left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2} \right), \\ R_{22}^2 &= R_0 \left(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2} \right) \end{aligned} \tag{4}$$

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$$d_{21}^2 = R_0\left(R_0 + r_0 - \sqrt{(R_0+r_0)^2 - d_0^2}\right), \quad d_{22}^2 = R_0\left(R_0 - r_0 - \sqrt{(R_0-r_0)^2 - d_0^2}\right), \tag{5}$$

$$r_{21}^2 = (R_0+r_0)^2 - d_0^2, \quad r_{22}^2 = (R_0-r_0)^2 - d_0^2\,. \tag{6}$$

Also, it is easy to check that

$$R_{21}^2\, d_{21}^2 = R_{22}^2\, d_{22}^2 = R_0^2 d_0^2,\ \ R_{21}^2 - d_{21}^2 = 2R_0 r_{21},\ \ R_{22}^2 - d_{22}^2 = 2R_0 r_{22}\,. \tag{7}$$

By some straightforward calculations we conclude from (1)–(3) that

$$R_0 = \frac{R_2^2 - d_2^2}{2r_2}, \quad d_0 = \frac{2R_2 r_2 d_2}{R_2^2 - d_2^2}, \tag{8}$$

$$r_0^2 = -(R_2^2 + d_2^2 - r_2^2) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}\right)^2 := \varphi(R_2, d_2, r_2)\,. \tag{9}$$

We will need these important formulæ frequently in the sequel.

Moreover, replace R_0, d_0, r_0 in (1), (2) and (3) respectively by R_{21}, d_{21}, r_{21}. Then the solution in R_2, d_2, r_2 of the transformed system is given by

$$\begin{aligned}
R_{211}^2 &= R_{21}\left(R_{21} + r_{21} + \sqrt{(R_{21}+r_{21})^2 - d_{21}^2}\right),\\
R_{212}^2 &= R_{21}\left(R_{21} - r_{21} + \sqrt{(R_{21}-r_{21})^2 - d_{21}^2}\right),\\
d_{211}^2 &= R_{21}\left(R_{21} + r_{21} - \sqrt{(R_{21}+r_{21})^2 - d_{21}^2}\right),\\
d_{212}^2 &= R_{21}\left(R_{21} - r_{21} - \sqrt{(R_{21}-r_{21})^2 - d_{21}^2}\right),\\
r_{211}^2 &= (R_{21}+r_{21})^2 - d_{21}^2\,,\\
r_{212}^2 &= (R_{21}-r_{21})^2 - d_{21}^2\,.
\end{aligned}$$

By repeating the above procedure we can take, e.g. the lengths R_{211}, d_{211}, r_{211} instead of the lengths R_0, d_0, r_0 in the system (1)–(3).

Let us remark here that in what follows, only R_{21}, d_{21}, r_{21} and $R_{211}, d_{211}, r_{211}$, will be considered throughout this article.

In [3] two conjectures are posed, which are equivalent to the following conjecture.

CONJECTURE. Let $F_n(R_0, d_0, r_0) = 0$ be the Fuss' relation for a bicentric n–gon, where one circle is inside the other. Then Fuss' relation $F_{2n}(R_2, d_2, r_2) = 0$ for the depending bicentric $2n$–gon can be obtained by taking

$$F_n\Big(\frac{R_2^2 - d_2^2}{2r_2};\ \frac{2R_2 r_2 d_2}{R_2^2 - d_2^2};\ \varphi(R_2, d_2, r_2)\Big) = 0,$$

compare (8)-(9). Conversely, starting with the Fuss' relation $F_{2n}(R_2, d_2, r_2) = 0$ one obtains $F_n(R_0, d_0, r_0) = 0$ by taking (4)-(6) into account.

We have to point out that testing the validity of this conjecture for different positive integers $n \geq 3$, we prove it for numerous values of n.

In this article it is shown that the achievements of Theorem A remain valid when one circle is not inside the other, that is, when instead of incircle there is the excircle. In this respect let us remark that Richolet [5], using some results which originate back to Jacobi [2], showed how certain relations valid for bicentric $2n$–gons can be obtained from depending relations for bicentric n–gons. Richolet's mathematical tools involve elliptic functions. However, here we expose a method (rather elementary one) using Theorem A, to deduce some equations for bicentric $2n$-gon by adequate relations for bicentric n–gon.

2 Bicentric n-gons and $2n$-gons with Excircle

Generally speaking in the case when the bicentric n-gon has excircle (instead of incircle), very difficult calculations could appear. Therefore we shall restrict ourselves to the case when n is not large and use the following four well known facts concerning bicentric n-gons.

(i) If R_0, d_0, r_0 are lengths (in fact positive numbers) such that

$$d_0^2 - R_0^2 = 2r_0R_0, \quad d_0 + r_0 > R_0, \quad d_0 + R_0 > r_0, \tag{10}$$

then there is triangle $A_0B_0C_0$ such that

$R_0 =$ radius of circumcircle of $\Delta A_0B_0C_0$,
$r_0 =$ radius of excircle of $\Delta A_0B_0C_0$,
$d_0 =$ distance between centers of circumcircle and excircle.

(ii) If R_0, d_0, r_0 are lengths such that

$$R_0^2 - d_0^2 = 2d_0r_0, \quad d_0 + r_0 > R_0, \quad d_0 + R_0 > r_0, \tag{11}$$

then there is bicentric hexagon $A_0B_0C_0D_0E_0F_0$ such that

$R_0 =$ radius of circumcircle of $A_0B_0C_0D_0E_0F_0$,
$r_0 =$ radius of excircle of $A_0B_0C_0D_0E_0F_0$,
$d_0 =$ distance between centers of circumcircle and excircle.

(iii) If R_0, d_0, r_0 are lengths such that

$$R_0 = d_0, \quad 2R_0 > r_0, \tag{12}$$

then there is bicentric quadrilateral $A_0B_0C_0D_0$ such that

$R_0 =$ radius of circumcircle of $A_0B_0C_0D_0$,
$r_0 =$ radius of excircle of $A_0B_0C_0D_0$,
$d_0 =$ distance between centers of circumcircle and excircle.

(**iv**) If R_0, d_0, r_0 are lengths such that

$$R_0^4 - 2d_0^2R_0^2 - 4d_0r_0^2R_0 + d_0^4 = 0, \quad d_0 + r_0 > R_0, \quad d_0 + R_0 > r_0 \tag{13}$$

then there is bicentric octagon $A_0B_0C_0D_0E_0F_0G_0H_0$ such that

$$\begin{aligned} R_0 &= \text{ radius of circumcircle of } A_0B_0C_0D_0E_0F_0G_0H_0, \\ r_0 &= \text{ radius of excircle of } A_0B_0C_0D_0E_0F_0G_0H_0, \\ d_0 &= \text{ distance between centers of circumcircle and excircle.} \end{aligned}$$

Now we are ready to formulate our first main result.

THEOREM 1. Let R_0, d_0, r_0 be lengths such that

$$d_0 + R_0 > r_0 \text{ or } d_0 + r_0 > R_0\,. \tag{14}$$

Then respectively

$$d_{21} + R_{21} > r_{21} \text{ or } d_{21} + r_{21} > R_{21}. \tag{15}$$

PROOF. By direct calculation, using the relations (4)-(6) in Theorem A and by

$$R_{21}\,d_{21} = R_0d_0, \tag{16}$$

which follows from (7), we can write

$$\begin{aligned} &R_0 + d_0 > r_0 \Rightarrow (R_0 + d_0)^2 > r_0^2 \\ &\Leftrightarrow R_0^2 + 2R_0d_0 + d_0^2 > r_0^2 \\ &\Leftrightarrow 2R_0(R_0 + r_0) + 2R_0d_0 > R_0^2 + 2R_0r_0 + r_0^2 - d_0^2 \\ &\Leftrightarrow d_{21}^2 + 2d_{21}R_{21} + R_{21}^2 > r_{21}^2 \\ &\Leftrightarrow d_{21} + R_{21} > \pm\, r_{21}. \end{aligned} \tag{17}$$

Now, bearing in mind that our model contains the excircle, we easily drop the negative sign on the last inequality, completing the proof of the first statement in (15).

Next, assuming $d_0+r_0 > R_0$, once more with the aid of (4)-(6), (16) and the excircle properties, we easily find that

$$\begin{aligned} &d_0 + r_0 > R_0 \quad \text{ or } \quad r_0 > R_0 - d_0 \Rightarrow r_0^2 > (R_0 - d_0)^2 \\ &\Leftrightarrow R_0^2 + 2R_0r_0 + r_0^2 - d_0^2 > 2R_0(R_0 + r_0) - 2R_0d_0 \\ &\Leftrightarrow r_{21}^2 > 2R_0(R_0 + r_0) - 2R_0d_0 \\ &\Leftrightarrow r_{21}^2 > R_{21}^2 + d_{21}^2 - 2R_{21}d_{21} \\ &\Rightarrow d_{21} + r_{21} > \pm\, R_{21}\,. \end{aligned} \tag{18}$$

Cancelling the negative sign on the last inequality, we obtain the proof.

THEOREM 2. Let R_0, d_0, r_0 be the lengths such that (10) holds, that is,

$$d_0^2 - R_0^2 = 2r_0R_0, \quad d_0 + r_0 > R_0, \quad d_0 + R_0 > r_0.$$

Then there is bicentric hexagon $A_0B_0C_0D_0E_0F_0$ such that

$$\begin{aligned} R_{21} &= \text{ radius of circumcircle of } A_0B_0C_0D_0E_0F_0, \\ r_{21} &= \text{ radius of excircle of } A_0B_0C_0D_0E_0F_0, \\ d_{21} &= \text{ distance between centers of circumcircle and excircle.} \end{aligned}$$

PROOF. According to (11), we have to prove

$$R_{21}^2 - d_{21}^2 = 2d_{21}r_{21}\,. \tag{19}$$

To do this, we bear in mind the first relations in (4)-(6). Then

$$\begin{aligned} & d_0^2 - R_0^2 = 2r_0R_0 \\ & \Leftrightarrow r_0^2 = (R_0+r_0)^2 - d_0^2 \\ & \Rightarrow R_0 = R_0 + r_0 - \sqrt{(R_0+r_0)^2 - d_0^2} \\ & \Leftrightarrow R_0^2 = d_{21}^2 \\ & \Leftrightarrow R_0^2[(R_0+r_0)^2 - d_0^2] = d_{21}^2\, r_{21}^2 \\ & \Leftrightarrow 2R_0\sqrt{(R_0+r_0)^2 - d_0^2} = \pm\, 2d_{21}r_{21} \\ & \Rightarrow R_{21}^2 - d_{21}^2 = 2d_{21}\, r_{21}\,. \end{aligned} \tag{20}$$

Here $R_0^2 = d_{21}^2$ can be concluded by the fact that only the lengths $(\cdot)_1$ is considered, and that there is the excircle case; while the last equality is obtained by rejecting the negative sign in the previous equality.

In the following examples, in calculating tangent lengths for $A_1 \dots A_n$, we will apply the well–known formula

$$(t_2)_{1,2} = \frac{(R^2-d^2)t_1 \pm \sqrt{D}}{r^2+t_1^2}, \tag{21}$$

where

$$D = t_1^2(R^2-d^2)^2 + (r^2+t_1^2)\big[4d^2R^2 - r^2t_1^2 - (R^2+d^2-r^2)^2\big]\,,$$

and R, r, d denote the radii of circumcircle, incircle and the distance between centers of these two circles respectively. If t_1 is given, then the consequent t_2's role will be played by t_{21} or t_{22}. The same relation is valid when instead of incircle the excircle appears.

Of course, if $A_1 \dots A_n$ is a bicentric n-gon, where instead of incircle there is excircle, then tangent-length t_i is given by $t_i = |A_iP_i|$, where P_i is tangent point of the line $|A_iA_{i+1}|$ and the excircle.

EXAMPLE 1. Let R_0, d_0, r_0 be such that (10) holds, that is,

$$R_0 = 2, \quad d_0 = 5, \quad r_0 = 5.25$$

and $t_1 = 4$. Then for corresponding triangle $A_0B_0C_0$ we have

$$t_2 = -3.58041..., \quad t_3 = -0.27611..., \quad t_4 = t_1,$$

noting that $\sum_{i=1}^{3} \arctan\big(t_i/r_0\big) = 0$. In the above exposed results negative t's appear. To this respect consult [4, p. 98].

For corresponding bicentric hexagon $A_0B_0C_0D_0E_0F_0$, where

$$R_{21} = 5, \quad d_{21} = 2, \quad r_{21} = 5.25$$

and $t_1 = 4$ we have

$$t_2 = 0.27611..., t_3 = -3.58041..., t_4 = -t_1, t_5 = -t_2,$$
$$t_6 = -t_3, t_7 = t_1, \sum_{i=1}^{6} \arctan\big(t_i/r_{21}\big) = 0\,.$$

For corresponding bicentric 12-gon where

$$R_{211} = 10.07546..., \quad d_{211} = 0.99251..., \quad r_{211} = 10.05298...$$

and $t_1 = 4$ we have

$$t_2 = 2.25780..., t_3 = 0.27611..., t_4 = -1.70889..., t_5 = -3.58041...,$$
$$t_6 = -4.61236..., t_7 = -t_1, t_8 = -t_2, t_9 = -t_3,$$
$$t_{10} = -t_4, t_{11} = -t_5, t_{12} = -t_6, t_{13} = t_1,$$

$$\sum_{i=1}^{12} \arctan\big(t_i/r_{211}\big) = 0\,.$$

At this moment let us remark that the same t_1 can be taken for bicentric n-gon and corresponding bicentric $2n$-gon since there holds the relation

$$\sqrt{\big(R_{21} + d_{21}\big)^2 - r_{21}^2} = \sqrt{(R_0 + d_0)^2 - r_0^2}\,.$$

In this respect we point out that the largest tangent that can be drawn from circumcircle to excircle is given by $\sqrt{(R_0 + d_0)^2 - r_0^2}$. The least tangent does not exist because the intersection of circumcircles and excircles is nonempty.

THEOREM 3. Let R_0, d_0, r_0 be such that (12) holds, that is,

$$R_0 = d_0, \quad r_0 < 2R_0.$$

Then there is bicentric octagon $A_0B_0C_0D_0E_0F_0G_0H_0$ such that

$$R_{21} = \text{ radius of circumcircle of } A_0B_0C_0D_0E_0F_0G_0H_0,$$
$$r_{21} = \text{ radius of excircle of } A_0B_0C_0D_0E_0F_0G_0H_0,$$
$$d_{21} = \text{ distance between centers of circumcircle and excircle,}$$

where for calculating R_{21}, r_{21} and d_{21} we use relations given by (4), (5), (6) and $R_0 = d_0$, $r_0 < 2R_0$.

PROOF. According to (13), we have to prove that

$$R_{21}^4 - 2d_{21}^2 R_{21}^2 - 4d_{21} r_{21}^2 R_{21} + d_{21}^4 = 0\,. \tag{22}$$

It is not difficult to find that

$$\begin{aligned} R_{21}^4 + d_{21}^4 &= 4R_0^2(R_0 + r_0)^2 - 2R_0^2 d_0^2, \\ -2d_{21}^2 R_{21}^2 &= -2R_0^2 d_0^2, \\ -4d_{21} R_{21} r_{21}^2 &= -4R_0 d_0[(R_0 + r_0)^2 - d_0^2]; \end{aligned}$$

now, since $d_0 = R_0$ we easily deduce (22).

EXAMPLE 2. Let R_0, d_0, r_0 be such that (12) holds, that is,

$$R_0 = 5, \quad d_0 = 5, \quad r_0 = 6$$

and $t_1 = 4$. Then for corresponding bicentric quadrilateral $A_0B_0C_0D_0$ we have

$$t_2 = -5.76461..., t_3 = -t_1, t_4 = -t_2, t_5 = t_1, \; \sum_{i=1}^{4} \arctan\big(t_i/r_0\big) = 0.$$

For corresponding bicentric octagon $A_0B_0C_0D_0E_0F_0G_0H_0$ where

$$R_{21} = 10.19753..., d_{21} = 2.45157..., r_{21} = 9.79795...$$

and $t_1 = 4$ we have

$$\begin{aligned} &t_2 = -0.87131..., t_3 = -5.76461..., t_4 = -7.86985..., t_5 = -t_1, \\ &t_6 = -t_2, t_7 = -t_3, t_8 = -t_4, t_9 = t_1, \end{aligned}$$

$$\sum_{i=1}^{8} \arctan\big(t_i/r_{21}\big) = 0\,.$$

For the corresponding bicentric 16-gon where

$$R_{211} = 20.15617..., d_{211} = 1.24031..., r_{211} = 19.84463...$$

and $t_1 = 4$ we have

$$\begin{aligned} &t_2 = 1.53118..., t_3 = -0.87131..., t_4 = -3.31870..., t_5 = -5.76461..., \\ &t_6 = -7.60826..., t_7 = -7.86985..., t_8 = -6.36971..., t_9 = -t_1, \\ &t_{10} = -t_2, t_{11} = -t_3, t_{12} = -t_4, t_{13} = -t_5, \\ &t_{14} = -t_6, t_{15} = -t_7, t_{16} = -t_8, t_{17} = t_1, \end{aligned}$$

$$\sum_{i=1}^{16} \arctan\big(t_i/r_{211}\big) = 0\,.$$

REMARK. Concerning the Conjecture posed previously, we can make the following remark. Let R_0, d_0, r_0 be any given lengths such that there is a bicentric n-gon $A_1 \dots A_n$ where

$$\begin{aligned} R_0 &= \text{ radius of circumcircle of } A_1 \dots A_n, \\ r_0 &= \text{ radius of excircle of } A_1 \dots A_n, \\ d_0 &= \text{ distance between centers of circumcircle and excircle,} \end{aligned}$$

and $d_0 + r_0 > R_0$ and $d_0 + R_0 < r_0$. Then there is a bicentric $2n$-gon $B_1 \dots B_{2n}$ such that

$$\begin{aligned} R_{21} &= \text{ radius of circumcircle of } B_1 \dots B_{2n}, \\ r_{21} &= \text{ radius of excircle of } B_1 \dots B_{2n}, \\ d_{21} &= \text{ distance between centers of circumcircle and excircle;} \end{aligned}$$

to obtain R_{21}, r_{21} and d_{21} we apply (4)-(6) respectively.

The Conjecture is proved for $n = 3$ and $n = 4$, see Theorems 1, 2 and 3. For $n = 5, 6, 7, 8$ we test the Conjecture by many tricky examples; however, the Conjecture remains valid in all those cases. So, we are asking for the general proof, whether our Conjecture is true for every given n.

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