Applied Mathematics E-Notes, 8(2008), 1-8 ©
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ISSN 1607-2510

Time Periodic Solutions To A Nonhomogeneous Dirichlet Periodic Problem*

Abderrahmane El Hachimi and Abdelilah Lamrani Alaoui†

Received 1 September 2006

Abstract

In this paper we are concerned with the existence of solutions for nonhomogeneous Dirichlet periodic problem associated with the following doubly nonlinear equation $\frac{\partial b(u)}{\partial t} - div(|\nabla u|^{p-2}\nabla u) = f(x,t,u)$. The results are obtained by using pseudomonotonicity arguments and an ϵ-version of Tartar's inequality.

1 Introduction

This paper is concerned with the existence of solutions of the following doubly nonlinear periodic elliptic-parabolic type problem

$$(\mathcal{P}) \begin{cases} \frac{\partial b(u)}{\partial t} - \triangle_p u = f(x,t,u) \ \text{ in } Q_T, \\ u = \varphi \ \text{ on } \Sigma_T, \\ b(u)(0) = b(u)(T) \ \text{ in } \Omega, \end{cases}$$

where $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$ is the so-called p-Laplacian operator, $1 < p < +\infty$, Ω is a regular and bounded subset of $\mathbb{R}^n$, $Q_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$, $\partial\Omega$ is the boundary of Ω and T is a positive real number. Besides its mathematical interest, the p-Laplacian appears in non-Newtonian fluids, in gazes infiltration in porous media, in glaciology and in some biological phenomena.

When $b = id_{\mathbb{R}}$ and $p \geq 2$, problem $(\mathcal{P})$ has been studied in [2] by Boldrini and Crema by using a fixed point argument. While in [8], the authors use subdifferential methods to deal with the case where $\triangle_p u$ is replaced by $\triangle g(u)$, $b = id_{\mathbb{R}}$ and f does not depend on u. We mention that there are many results concerning the existence of periodic solutions mainly for semilinear PDEs, by Angenent, Fiedler, Matano, Malett-Paret and so on. There is a good survey by Polacik [11]. In [8], the authors give existence and stabilization results in the case of an elliptic operator of Leray-Lions type with nonlinearities of natural growth in $|\nabla u|$. When $b \neq id_{\mathbb{R}}$, problem $(\mathcal{P})$ with general quasilinear elliptic operator instead of $\triangle_p$ has been treated by various authors (for example by linear methods as in [9]). A dynamical system approach has also been used in [5], [4] and [6], to deal with the asymptotic behavior of the problem $(\mathcal{P})$ with periodic condition replaced by initial data.

*Mathematics Subject Classifications: 35K15, 35K55, 35B10.

†UFR Mathématiques Appliquées et Industrielles, Faculté des Sciences, B.P. 20, El Jadida - Maroc.

2 Hypotheses and Definitions

We suppose the following hypotheses:

(H1) $b : \mathbb{R} \to \mathbb{R}$ is a nondecreasing locally Lipschitzian function such that $b(0) = 0$.

(H2) $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, i.e. $f(x, ., .)$ is continuous function for almost every $x \in \Omega$ on $\mathbb{R}^+ \times \mathbb{R}$, and $f(., t, \xi)$ is measurable on Ω for all $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}$, and satisfies $|f(x, t, s)| \leq a(|s|)$, for all $s \in \mathbb{R}$, and a.e. $(x, t) \in \Omega \times \mathbb{R}^+$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function.

(H3) f is a T-time periodic function i.e. for any $s \in \mathbb{R}$, and a.e. $(x, t) \in \Omega \times \mathbb{R}^+$ we have $f(x, t + T, s) = f(x, t, s)$.

(H4) $\varphi \in L^p(0, T; W^{1,p}(\Omega))$ is such that: $\frac{\partial \varphi}{\partial t} \in L^{p'}(\mathcal{Q}_T)$ and $\varphi(0) = \varphi(T)$ in $L^1(\Omega)$, where p' is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

(H5) There exist positive constants α and β such that $\alpha \leq \varphi(x, t) \leq \beta$ and $f(x, t, \beta) \leq 0 \leq f(x, t, \alpha)$ a.e. in $\mathcal{Q}_T$.

Recall that a function u is called a weak solution of problem $(\mathcal{P})$ if u is such that $u \in \mathcal{V}_0 + \varphi$, where $\mathcal{V}_0 := L^p(0, T; W_0^{1,p}(\Omega))$, $u \in L^\infty(\mathcal{Q}_T)$, $\frac{\partial b(u)}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, and for all $v \in \mathcal{V}_0$,

$$\int_0^T \left\langle \frac{\partial b(u)}{\partial t}, v \right\rangle + \int_{\mathcal{Q}_T} |\nabla u|^{p-2} \nabla u . \nabla v = \int_{\mathcal{Q}_T} f(x, t, u) v$$

and

$$\int_0^T \left\langle \frac{\partial b(u)}{\partial t}, v \right\rangle = - \int_{\mathcal{Q}_T} b(u) \frac{\partial v}{\partial t}$$

for any $v \in W^{1,1}(0, T; L^1(\Omega)) \cap \mathcal{V}_0$ with $v(0) = v(T)$ in $L^1(\Omega)$.

Let k be a nonnegative constant satisfying $-k + 1 \leq \alpha \leq \beta \leq k - 1$. We shall denote by $\mathcal{K}$ the set

$$\mathcal{K} := \{u \in \mathcal{V}_0 + \varphi \text{ such that } -k \leq u \leq k \text{ a.e. in } \mathcal{Q}_T\}$$

3 Main Result

The main result is the following.

THEOREM 3.1. Let us suppose that the conditions (H1)-(H5) are satisfied. Then there exists a solution u of $(\mathcal{P})$ such that: $b(\alpha) \leq b(u) \leq b(\beta)$ a.e. in $\mathcal{Q}_T$.

Before proving our main result, we have several remarks. First, as announced above, the method used here, which is based on pseudomonotonicity arguments combined with an ϵ-version of Tartar's inequality, will also work for similar hypothesis on the data in both parabolic and elliptic cases if the singular p-Laplacian is replaced by the regularized one

$$\Delta_p^\epsilon u := div(|\nabla u|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u. \tag{1}$$

This last operator can be viewed as the subdifferential of a fixed function [3].

Second, to our knowledge, the regularized p-Laplacian operator (1) has never been treated by pseudomonotonicity arguments.

Third, the information $u \in L^{\infty}(\mathcal{Q}_T)$ can be viewed as a consequence of a localization property of the approximate solutions. For this aim a comparison principle is needed. Note that in [4], a comparison principle is obtained with the regularity supplementary hypothesis:

$$\frac{\partial b(u)}{\partial t} \in L^1(\mathcal{Q}_T) \tag{2}$$

This hypothesis can be recuperated with additional hypothesis on b and on the initial data, in the case of initial value problems. For periodic problems, it becomes a regularity property on the whole solution.

We have opted here for the introduction of regular class of lower and upper solutions (constants upper and lower solutions).

Finally let us mention that the results of [2] obtained in the case of $p \geq 2$ can be extended to $1 < p < 2$ by replacing the singular p-Laplacian by Δ_p^{ϵ} in (1) and using the same fixed point argument and the ϵ-version of Tartar's inequality claimed here.

4 Proof of the Main Result

We shall use a regularization method: In the degenerate case $p \geq 2$, we study the sequence of problems

$$(\mathcal{P}_n)\begin{cases} \frac{\partial b_n(u)}{\partial t} - \triangle_p u = f(x,t,u) \text{ in } Q_T, \\ u = \varphi \text{ on } \Sigma_T, \\ b_n(u)(0) = b_n(u)(T) \text{ in } \Omega, \end{cases}$$

where b_n is sufficiently regular. $(\mathcal{P}_n)$ is equivalent to

$$(\widetilde{\mathcal{P}}_n)\begin{cases} \frac{\partial u}{\partial t} - div(\frac{1}{b_n'(u+\varphi)}F(\nabla(u+\varphi))) = \frac{1}{b_n'(u+\varphi)}f(x,t,u) \\ \qquad + \frac{b''(u+\varphi)}{(b_n'(u+\varphi))^2}|\nabla(u+\varphi)|^p - \frac{\partial \varphi}{\partial t} \text{ in } Q_T, \\ u = 0 \text{ on } \Sigma_T, \\ u(0) = u(T) \text{ in } \Omega, \end{cases}$$

where $F(\zeta) = |\zeta|^{p-2}\zeta, \forall \zeta \in \mathbb{R}^n$.

Note that in this case, existence is an immediate consequence of [7]. Hence, we shall focus our attention on the case $1 < p < 2$, which is more complicated than the degenerate one.

$(\mathcal{P}_n)$ is approximated by

$$(\mathcal{P}_{n,\epsilon})\begin{cases} \frac{\partial b_n(u)}{\partial t} - \triangle_p^{\epsilon} u = f(x,t,u) \text{ in } Q_T, \\ u = \varphi \text{ on } \Sigma_T, \\ b_n(u)(0) = b_n(u)(T) \text{ in } \Omega, \end{cases}$$

where $\triangle_p^\epsilon$ is defined by (1). $(\mathcal{P}_{n,\epsilon})$ is equivalent to

$$(\widetilde{\mathcal{P}}_{n,\epsilon})\begin{cases} \frac{\partial u}{\partial t} - div(\frac{1}{b'_n(u)}F_\epsilon \nabla u) = \frac{1}{b'_n(u)}f(x,t,u) \\ \quad + \frac{b''(u)}{(b'(u))^2}(|\nabla u|^2+\epsilon)^{\frac{p-2}{2}}|\nabla u|^2 \text{ in } Q_T, \\ u=\varphi \text{ on } \Sigma_T, \\ u(0)=u(T) \text{ in } \Omega. \end{cases}$$

LEMMA 4.1. Let $\widetilde{b}\in C(\mathbb{R})$ be such that, $0<m\le \widetilde{b}(s)\le M$ on $\mathbb{R}$ with positive real constants: m and M are nonnegative real constants. If $\widetilde{f}:\mathcal{Q}_T\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a Caratheodory function with $|\widetilde{f}(x,t,s,\zeta)|\le M$ a.e. $(x,t)\in\mathcal{Q}_T$ and $\forall(s,\zeta)\in\mathbb{R}^{n+1}$ and φ satisfies (H1)-(H5). Then there exists $u\in\mathcal{K}$ such that

$$(\mathcal{P}_{1,T})\begin{cases} \langle\frac{\partial u}{\partial t}, v-u\rangle + \int_{\mathcal{Q}_T}\widetilde{b}(u)F_\epsilon(\nabla u)\nabla(v-u) \\ \ge \int_{\mathcal{Q}_T}\widetilde{f}(x,t,u,\nabla u)(v-u) \text{ for all } v\in\mathcal{K}, \\ u(0)=u(T) \text{ in } \Omega. \end{cases}$$

PROOF. We consider the following sequence of problems:

$$(\mathcal{P}_{\eta,T})\begin{cases} \frac{\partial u}{\partial t} - div(\widetilde{b}(u)F_\epsilon(\nabla u)) + \frac{\beta(u)}{\eta} = \widetilde{f}(x,t,u,\nabla u) \text{ in } Q_T, \\ u=\varphi \text{ on } \Sigma_T, \\ u(0)=u(T) \text{ in } \Omega, \end{cases}$$

where

$$\beta(s)=\begin{cases} (s-k)^{p-1} & \text{if } s>k, \\ 0 & \text{if } |s|\le k, \\ -(-s-k)^{p-1} & \text{if } s<-k, \end{cases}$$

and $F_\epsilon(\zeta)=(|\zeta|^2+\epsilon)^{\frac{p-2}{2}}\zeta, \forall\zeta\in\mathbb{R}^n$. Let us seek a solution of

$$(\widetilde{\mathcal{P}}_{\eta,T})\begin{cases} \frac{\partial v}{\partial t}+\mathcal{A}v &= \frac{\partial\varphi}{\partial t} & \text{in } Q_T, \\ v &= 0 & \text{on } \Sigma_T, \\ v(0) &= v(T) & \text{in } \Omega, \end{cases}$$

where

$$\mathcal{A}(v) := -div(\widetilde{b}(v+\varphi)F_\epsilon(\nabla v+\varphi)) + \frac{\beta(v+\varphi)}{\eta} - \widetilde{f}(x,t,v+\varphi,\nabla(v+\varphi))$$

is defined on $\mathcal{V}_0$.

A solution v_η of $(\widetilde{\mathcal{P}}_{\eta,T})$ will yield a solution u_η of $(\mathcal{P}_{\eta,T})$ by taking $u_\eta=v_\eta+\varphi$. An existence result for $(\widetilde{\mathcal{P}}_{\eta,T})$ holds if,for example, the hypothesis of pseudomonotonicity, boundedness and coerciveness [10, p. 319] are satisfied for $\mathcal{A}$. The two last hypothesis are easily verified while the first one needs the following lemma.

LEMMA 4.2. (ϵ-Tartar Inequality) For all p in (1,2) there exist $c>0$ such that for all ζ and ξ in $\mathbb{R}^n$ we have:

$$c|\xi-\zeta|^p \le \{(F_\epsilon(\xi)-F_\epsilon(\zeta),\xi-\zeta)\}^{\frac{p}{2}}\{|\xi|^p+|\zeta|^p+\epsilon^{\frac{p}{2}}\}^{1-\frac{p}{2}}.$$

PROOF. By using Taylor's expansion of $f(t) := \varphi_\epsilon(\xi + t(\zeta - \xi))$, on $[0,1]$ where:

$$\begin{aligned} \varphi_\epsilon : \mathbb{R}^n &\to \mathbb{R} \\ a &\to (|a|^2 + \epsilon)^{\frac{p}{2}}, \end{aligned}$$

we obtain the desired result.

Note that φ_ϵ is a convex function because of its continuity and mid-convexity. Applying Lemma 4.2 with $\xi = \nabla v_n$ and $\zeta = \nabla v$ where $(v_n)_n$ is a sequence such that $v_n \rightharpoonup v$ in $\mathcal{V}_0$ and $\frac{\partial v_n}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}$ and $v_n(0) = v_n(T)$, we have $v_n \to v$. Then pseudomonotonicity for $\mathcal{A}$ is established and existence of solution to $(\mathcal{P}_{\eta,T})$ is guaranteed. Let us now pass to the limit in $\eta \to 0^+$. First we collect some a priori estimates.

LEMMA 4.3. We have

$$\left(\frac{\beta u_\eta}{\eta}\right)_\eta \text{ is bounded in } L^{p'}(\mathcal{Q}_T). \tag{3}$$

$$(u_\eta)_\eta \text{ is bounded in } L^p(0,T;W^{1,p}(\Omega)). \tag{4}$$

$$\left(\frac{u_\eta}{\partial t}\right)_\eta \text{ is bounded in } L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{5}$$

PROOF. Taking $(u_\eta - k)^+$ as a test function in the equation of $(\mathcal{P}_{\eta,T})$, we have

$$\int_{\mathcal{Q}_T} \frac{[(u-k)^+]^p}{\eta} \le c \int_{\mathcal{Q}_T} (u-k)^+.$$

Then $\left(\frac{[(u-k)^+]^{p-1}}{\eta}\right)_\eta$ is bounded in $L^{p'}(\mathcal{Q}_T)$. In a similar way, $\left(\frac{[(u+k)^-]^{p-1}}{\eta}\right)_\eta$ is bounded in $L^{p'}(\mathcal{Q}_T)$. Then (3) is obtained. Moreover,

$$(u_\eta)_\eta \text{ is bounded in } L^p(\mathcal{Q}_T). \tag{6}$$

By using $v_\eta = u_\eta - \varphi$ as a test function in the equation of $(\widetilde{\mathcal{P}}_{\eta,T})$, we obtain

$$\int_{\mathcal{Q}_T} (|\nabla u_\eta|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\eta|^2 \le c + c\left(\int_{\mathcal{Q}_T} |\nabla u_\eta|^p\right)^{\frac{1}{p}}. \tag{7}$$

but

$$\int_{\mathcal{Q}_T} |\nabla u_\eta|^p = \int_{\mathcal{Q}_T} (|\nabla u_\eta|^2 + \epsilon)^{\frac{p(p-2)}{4}} |\nabla u_\eta|^p (|\nabla u_\eta|^2 + \epsilon)^{\frac{p(2-p)}{4}}.$$

Hence, by using Young's inequality we get

$$\int_{\mathcal{Q}_T} |\nabla u_\eta|^p \le c(\delta) \int_{\mathcal{Q}_T} (|\nabla u_\eta|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\eta|^2 + \delta \int_{\mathcal{Q}_T} (|\nabla u_\eta|^2 + \epsilon)^{\frac{p}{2}}.$$

For δ sufficiently small, we obtain

$$\int_{\mathcal{Q}_T} |\nabla u_\eta|^p \le c \int_{\mathcal{Q}_T} (|\nabla u_\eta|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u|^2 + c. \tag{8}$$

Now, as can easily be deduced, (6), (7) and (8) lead to (4) and (5). Thanks to relations (4) and (5), there exist $u \in L^p(0,T;W^{1,p}(\Omega))$ such that for a subsequence denoted again by $(u_\eta)_\eta$ we have $u_\eta \rightharpoonup u$ in $L^p(0,T;W^{1,p}(\Omega))$ and $u_\eta \to u$ in $L^p(\mathcal{Q}_T)$ and a.e. in $\mathcal{Q}_T$ when $\eta \to 0^+$. By (3), it follows that $-k \le u \le k$ a.e. in $\mathcal{Q}_T$ and since β is monotone, then u is a solution of $(\mathcal{P}_{1,T})$ (convergence $u_\eta \to u$ in $L^p(0,T;W^{1,p}(\Omega))$, can be obtained in similar way as in the proof of pseudomonotonicity of $\mathcal{A}$).

Let us now prove Theorem 3.1. Let$(b_n)_n$ be a sequence of $C^\infty(\mathbb{R})$ such that $\frac{1}{n} \le b_n^{'}(s) \le C_n,\ 0 \le b^{''}(s) \le C_n,\ b_n(0) = 0$ and $b_n \to b$ in $C_{loc}(\mathbb{R})$. We have the following.

LEMMA 4.4. For each $\epsilon > 0$ and $n \in \mathbb{N}^*$, there exists a solution $u \in \mathcal{K}$ of $(\mathcal{P}_{n,\epsilon})$.

PROOF. For $\delta > 0$, we define:

$$\widetilde{f}_\delta(x,t,u,\nabla u) := \frac{\widehat{f}(x,t,u,\nabla u)}{1+\delta|\widehat{f}(x,t,u,\nabla u)|}$$

where

$$\widehat{f}(x,t,u,\nabla u) = \frac{1}{b_n'(S(u))} f(x,t,S(u)) + \frac{b''S(u)}{(b'(S(u)))^2}(|\nabla S(u)|^2+\epsilon)^{\frac{p-2}{2}}|\nabla S(u)|^2,$$

with

$$S(u)(x,t) = \begin{cases} \beta & \text{if } u(x,t) > \beta, \\ u(x,t) & \text{if } \alpha \le u(x,t) \le \beta, \\ \alpha & \text{if } u(x,t) < \alpha. \end{cases}$$

By using Lemma 4.1, there exists a solution u_δ of

$$(\mathcal{P}_{\delta,T}) \begin{cases} \left\langle \frac{\partial u}{\partial t}, v-u \right\rangle + \int_{\mathcal{Q}_T} \frac{1}{b_n'(S(u))} F_\epsilon(\nabla u)\nabla(v-u) \\ \quad \ge \int_{\mathcal{Q}_T} \widetilde{f}_\delta(x,t,u,\nabla u)(v-u) \text{ for all } v \in \mathcal{K}, \\ \hfill u(0) = u(T) \text{ in } \Omega. \end{cases}$$

By choosing $v = u - t(\psi_\lambda(u) - \psi_\lambda(\varphi))$ for $t > 0$ sufficiently small and $\psi_\lambda(s) = se^{\lambda s^2}$, for $\lambda > 0$ to be fixed later, we obtain

$$\int_{\mathcal{Q}_T} \left[\frac{1}{C_n}\psi_\lambda^{'}(u) - c\psi_\lambda(u)\right]|\nabla u|^p \le c,$$

with λ sufficiently large, u is bounded in $L^p(0,T,W^{1,p}(\Omega))$, and by choosing separately $v = u - t\theta^+$ and $v = u + t\theta^-$ with $\theta \in L^\infty(\mathcal{Q}_T) \cap \mathcal{V}_0$ we obtain that $(\frac{\partial u_n}{\partial t})_n$ is bounded in $\mathcal{V}_0' + L^1(\mathcal{Q}_T)$. Let us now consider the following problem

$$(\mathcal{P}_{\delta,\eta,T}) \begin{cases} \frac{\partial u}{\partial t} - div(\frac{1}{b_n'(S(u))}F_\epsilon(\nabla u) + \frac{\beta(u)}{\eta} = \widetilde{f}_\delta(x,t,u,\nabla u) \text{ in } Q_T, \\ \hfill u = \varphi \text{ on } \Sigma_T, \\ \hfill u(0) = u(T) \text{ in } \Omega, \end{cases}$$

with indices δ and δ' . By using $\psi_\lambda(u_\delta - u_{\delta'})$ as a test function and applying Aubin's compactness theorem, the monotonicity of β and Lemma 4.2, we obtain for λ chosen sufficiently large that $u_\delta \to u$ in $L^p(0,T,W^{1,p}(\Omega))$. The existence of a solution to

$$(\mathcal{P}_{\epsilon,n,T})\begin{cases} \left\langle \frac{\partial u}{\partial t}, v-u\right\rangle + \int_{\mathcal{Q}_T} \frac{1}{b_n'(S(u))} F_\epsilon(\nabla u)\nabla(v-u) \\ \geq \int_{\mathcal{Q}_T} \widehat{f}(x,t,u,\nabla u)(v-u), \text{ for all } v \in \mathcal{K}, \\ \hfill u(0)=u(T) \text{ in } \Omega, \end{cases}$$

is now proved. The localization $\alpha \leq u \leq \beta$ can easily be shown and our result is proved.

Passing to the limit in $\epsilon \to 0$ and $n \to +\infty$ is almost the same as in [1] or [6].

Acknowledgment. The authors would like to thank the referees for useful remarks and for pointing out [11] to them.

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