Applied Mathematics E-Notes, 8(2008), 223-230 © ISSN 1607-2510
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A Note On The Unique Solvability Of An Inverse Problem With Integral Overdetermination*

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Received 28 June 2006

Abstract

We discuss the unique solvability of an inverse problem for parabolic equation with an integral overdetermination condition.

1 Introduction

In this paper we study the unique solvability of the inverse problem of determining a pair of functions $\{u, f\}$ satisfying the equation

$$u_t - \Delta u + \sum_{i=1}^{n} b_i(x)u_{x_i} + \alpha u = f(t)g(x,t),\ (x,t) \in Q_T \equiv \Omega \times (0,T), \tag{1}$$

the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2}$$

the boundary condition

$$u(x,t) = 0,\ x \in \partial\Omega, \quad t \in (0,T) \tag{3}$$

and the overdetermination condition

$$\int_\Omega u(x,t)w(x)dx = \xi(t), \quad t \in (0,T), \tag{4}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$. The functions g, w, u_0, ξ and the positive constant α are given while $\{u, f\}$ is unknown. Additional information about the solution to the inverse problem is given in the form of integral overdetermination condition (4).

There are some papers devoted to the study of existence and uniqessness of solutions of inverse problems for various parabolic equations with unknown source functions. Inverse problems of determining the right-hand side of a parabolic equation

*Mathematics Subject Classifications: 45D05, 49K22.

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under a final overdetermination condition were studied in papers [1, 2, 3, 4]. The existence of smooth solutions of the corresponding inverse problem for parabolic equations with smooth coefficients was studied in [3, 4, 7]. Properties of increased smoothness for the inverse problem for parabolic equation with variable coefficients, under a condition on differentiability with respect to time of the source function are investigated in [10]. The generalized solutions and asymptotic stability of the inverse problem for parabolic equations were considered in [2, 3, 5, 8, 9, 10].

Let us introduce certain notations used below. We set

$$g_1(t) = \int_\Omega g(x,t)w(x)dx, Q_T = \Omega \times (0,T). \tag{5}$$

The spaces $W_2^1(\Omega)$, $rW_2^1(\Omega)$, $C(0,T;L_2(\Omega))$ and $W_2^{2,1}(Q_T)$ with corresponding norms are understood as follows: (see [6]) the Banach Space $W_2^1(\Omega)$ consists of all functions from $L_2(\Omega)$ having all weak derivatives of the first order that are square integrable over Ω with norm

$$\|u\|_{2,\Omega}^{(1)} = \left(\|u\|_{2,\Omega}^2 + \|u_x\|_{2,\Omega}^2\right)^{1/2}.$$

By $rW_2^1(\Omega)$, we denote the Banach function spaces obtained by the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W_2^1(\Omega)$. The space $C((0,T);L_2(\Omega))$ comprises of all continuous functions on $(0,T)$ with values in $L_2(\Omega)$. The corresponding norm is given by

$$\|u\|_{C((0,T);L_2(\Omega))} = \max_{(0,T)} \|u(t)\|_{2,\Omega} < \infty.$$

Let us also introduce the Sobolev space $W_2^{2,1}(Q_T)$ of functions $u(x,t)$ with finite norm

$$\|u\|_{W_2^{2,1}(Q_T)} = \left(\|u\|_{L_2(Q_T)}^2 + \|D_t u\|_{L_2(Q_T)}^2 + \sum_{j=1}^{2} \|D_x^j u\|_{L_2(Q_T)}^2\right)^{1/2}$$

where

$$\|u\| \equiv \|u\|_{L_2(\Omega)}$$

for $u(x) \in L_2(\Omega)$ and we denote by θ the constant from the Poincare's inequality

$$\|u\| \le \theta \|\nabla u\| \tag{6}$$

which is valid for each $u(x) \in rW_2^1(\Omega)$ and $\theta = \theta(\Omega, n) > 0$. We note that the weighted arithmetic-geometric mean inequality is:

$$2|ab| \le \epsilon a^2 + \epsilon^{-1} b^2 \tag{7}$$

for $\epsilon > 0$. Here

$$\|\nabla u\| = \left(\int_\Omega \sum_{i=1}^n u_{x_i}^2 dx\right)^{1/2} \text{ and } \|\Delta u\| = \left(\int_\Omega \sum_{i,j=1}^n u_{x_i x_j}^2 dx\right)^{1/2}.$$

We shall assume that the functions appearing in the data for the problem are measurable and satisfy the following conditions;

$$g \in C((0,T); L_2(\Omega)), w \in W_2^2(\Omega) \cap rW_2^1(\Omega), \varphi \in W_2^1(0,T),$$
$$\|g(x,t)\| \le K_g, |g_1(t)| \ge g_0 \equiv \text{constant} > 0 \quad \text{for} \quad t \in (0,T), \tag{8}$$
$$u_0 \in W_2^2(\Omega) \cap rW_2^1(\Omega), \xi \in W_2^1(0,T), \int_\Omega u_0(x)w(x)dx = \xi(0),$$

where

$$K_g, g_0, B_0 = esssup \left\{ \sum_{i=1}^n b_i^2(x) \right\}^{1/2}$$

are positive constants.

We multiply the equation (1) by w, integrate by parts over Ω and assume that $(\nabla u, w)_\Omega$ vanishes. Then, from (4) and (5) we obtain the relation

$$f(t) = \frac{1}{g_1} \left\{ \xi'(t) + \alpha\xi(t) + \int_\Omega \nabla u \nabla w dx + \int_\Omega \sum_{i=1}^n b_i u_{x_i} w dx \right\} \tag{9}$$

where both sides are treated as elements of $L_2(0,T)$.

2 Unique solvability of the inverse problem

We first state the following

DEFINITION 1. A pair of functions $\{u, f\}$ is said to be a generalized solution of the inverse problem (1)-(4) if $u \in W_{2,0}^{2,1}(Q_T)$, $f \in L_2(0,T)$ and all of the relations (1)-(4) are satisfied.

We seek a solution of the original inverse problem as $\{u, f\} = \{z, f\} + \{\nu, 0\}$ where ν is the solution of the direct problem

$$\nu_t - \Delta\nu + \sum_{i=1}^n b_i(x)\nu_{x_i} + \alpha\nu = 0, \quad (x,t) \in Q_T, \tag{10}$$

$$\nu(x,0) = u_0(x), \quad x \in \Omega, \tag{11}$$

$$\nu(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{12}$$

while the pair $\{z, f\}$ is the solution of the inverse problem

$$z_t - \Delta z + \sum_{i=1}^n b_i(x) z_{x_i} + \alpha z = f(t)g(x,t), \quad (x,t) \in Q_T, \tag{13}$$

$$z(x,0) = 0, \quad x \in \Omega, \tag{14}$$

$$z(x,t) = 0, x \in \partial\Omega, \quad t \in (0,T), \tag{15}$$

$$\int_{\Omega} z(x,t)w(x)dx = \varphi(t), \quad t \in (0,T) \tag{16}$$

where $\varphi(t) = \xi(t) - \int_{\Omega} \nu(x,t)w(x)dx$.

DEFINITION 2. By an energy solution of the problem (10)-(12) we mean a function $\nu \in W_2^{2,1}(Q_T)$ satisfying the corresponding integral identity (see [4]).

By Ref. [4], an energy solution of problem (10)-(12) exists and is unique. Let us remark that the study of the unique solvability of problem (1)-(4) is equivalent to that of the unique solvability of the inverse problem (13)-(16).

Now, our aim is to derive a linear second order equation of the Volterra type for the coefficient f over the space $L_2(0,T)$. The well-founded choice of a function f from the space $L_2(0,T)$ may be of help in achieving this aim. Substitution into (13) motivates that the system (13)-(15) serves as a basis finding the function $z \in W_{2,0}^{2,1}(Q_T)$ as a unique solution of the direct problem (13)-(15). The correspondence between f and z may be viewed as one possible way of specifying the linear operator

$$A : L_2(0,T) \longmapsto L_2(0,T) \tag{17}$$

with the values

$$(Af)(t) = \frac{1}{g_1}\left\{\int_{\Omega} \nabla z \nabla w dx + \int_{\Omega} \sum_{i=1}^{n} b_i z_{x_i} w dx\right\} \tag{18}$$

where $g_1(t) = \int_{\Omega} g(x,t)w(x)dx$.

In this view, it is reasonable to refer to the linear equation of the second kind for the function f over the space $L_2(0,T)$

$$f = Af + \psi \tag{19}$$

where $\psi = \frac{\varphi' + \alpha\varphi}{g_1}$.

THEOREM 1. Suppose the input data of the inverse problem (13)-(16 satisfies (8). Then the following assertions are valid: (i) if the inverse problem (13)-(16) is solvable, then so is equation (19), and (ii) if equation (19) possesses a solution and the compatibility condition

$$\varphi(0) = 0 \tag{20}$$

holds, then there exist a solution of the inverse problem (13)-(16).

PROOF. (i) Assume that the inverse problem (13)-(16) is solvable. We denote its solution by $\{z, f\}$. Multiplying both sides of (13) by the function w scalarly in $L_2(\Omega)$ we obtain the relation

$$\frac{d}{dt}\int_{\Omega} zwdx + \int_{\Omega} \nabla z \nabla w dx + \int_{\Omega} \sum_{i=1}^{n} b_i(x) z_{x_i} w dx + \alpha \int_{\Omega} zwdx = f(t)g_1(t). \tag{21}$$

With (16) and (18), it follows from (21) that $f = Af(\varphi' + \alpha\varphi)/g_1$. This means that f solves equation (19).

(ii) By the assumption, equation (19) has a solution in the space $L_2(0,T)$, say f. When inserting this function in (13), the resulting relations (13)-(15) can be treated as a direct problem having a unique solution $z \in W_{2,0}^{2,1}(Q_T)$.

Let us show that the function z satisfies also the integral overdetermination condition (16). Equation (12) yields

$$\frac{d}{dt}\int_\Omega zwdx + \int_\Omega \nabla z \nabla w dx + \int_\Omega \sum_{i=1}^n b_i(x) z_{x_i} w dx + \alpha \int_\Omega zwdx = f(t) g_1(t). \tag{22}$$

On the other hand, being a solution to equation (19), the function z is subject to relation

$$\varphi'(t) + \alpha\varphi(t) + \int_\Omega \nabla z \nabla w dx + \int_\Omega \sum_{i=1}^n b_i(x) z_{x_i} w dx = f(t) g_1(t). \tag{23}$$

Subtracting equation (22) from equation (23), we get

$$\frac{d}{dt}\int_\Omega zwdx + \alpha \int_\Omega zwdx = \varphi'(t) + \alpha\varphi(t).$$

Integrating the preceding differential equation and taking into account the compatibility condition (20), we find out that the function z satisfies the overdetermination condition (16) and the pair of functions $\{z, f\}$ is a solution of the inverse problem (13)-(16). This completes the proof of the theorem.

Now, it will be sensible to touch upon the properties of the operator A. The symbol A^s $(s = 1, 2, ...)$ refers to the s-th degree of the operator A.

LEMMA 1. Let the condition (8) hold. Then there exist a positive integer s_0 for which A^{s_0} is a contracting operator in $L_2(0,T)$.

PROOF. Obviously, (18) yields the estimate

$$\|Af\|_{2,(0,t)} \le \frac{K_w}{g_0}\left(\int_0^t \|\nabla z(.,\tau)\|_{2,\Omega}^2 d\tau\right)^{1/2}, 0 \le t \le T \tag{24}$$

where $K_w = \|\nabla w\|_{2,\Omega} + B_0 \|w\|_{2,\Omega}$. Multiplying both sides of (13) by z scalarly in $L_2(\Omega)$ and integrating the resulting expressions by parts, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\|z(.,t)\|^2 + \|\nabla z(.,t)\|^2 + \int_\Omega \sum_{i=1}^n b_i(x) z_{x_i} z dx + \alpha \|z(.,t)\|^2 = f(t)\int_\Omega gzdx,$$

and using Cauchy's, Poincare' and Young's inequalities we get the relation

$$\frac{1}{2}\frac{d}{dt}\|z(.,t)\|^2 + \left(1 - \frac{\theta}{2}(\delta_1+\delta_2) - \frac{B_0}{2\delta_1}\right)\|\nabla z(.,t)\|^2 + \alpha\|z(.,t)\|^2 \le \frac{K_g^2}{2\delta_2}|f(t)|^2. \tag{25}$$

Choosing $\delta_1, \delta_2 > 0$ such that $\gamma = (1 - \frac{\theta}{2}(\delta_1+\delta_2) - \frac{\mu_1}{2\delta_1}) > 0$ and integrating (25) from 0 to t, with (14) we obtain

$$\frac{1}{2}\|z(.,t)\|^2 + \gamma\int_0^t \|\nabla z(.,\tau)\|^2 d\tau + \alpha\int_0^t \|z(.,\tau)\|^2 d\tau \le \eta \int_0^t |f(\tau)|^2 d\tau \tag{26}$$

where $\eta = K_g^2/\delta_2 > 0$. Omitting some terms on the left-hand side (26) leads to

$$\int_0^t \|\nabla z(.,\tau)\|^2 d\tau \leq \frac{\eta}{\gamma} \int_0^t |f(\tau)|^2 d\tau. \tag{27}$$

It follows from (24) and (27), the estimate

$$\|Af\|_{2,(0,t)} \leq \frac{K_w \eta^{1/2}}{g_0 \gamma^{1/2}} \left(\int_0^t |f(\tau)|^2 d\tau \right)^{1/2}, 0 \leq t \leq T. \tag{28}$$

It is evident that for any positive integer s the s-th degree of the operator A can be defined in a natural way. By mathematical induction, (28) gives

$$\|A^s f\|_{2,(0,T)} \leq \frac{K_w^s \eta^{s/2}}{g_0^s \gamma^{s/2}} \|f\|_{2,(0,T)}, \ s = 1, 2, ... \tag{29}$$

It follows from the foregoing that there exists a positive integer $s = s_0$ such that

$$\frac{K_w^{s_0} \eta^{s_0/2}}{g_0^{s_0} \gamma^{s_0/2}} < 1. \tag{30}$$

Inequality (30) provides support for the view that the linear operator A^{s_0} is a contracting mapping on $L_2(0,T)$ and completes the proof of the lemma.

Regarding the unique solvability of the inverse problem concerned, the following result could be useful.

THEOREM 2. Let (8) and the compatibility condition (20) hold. Then the following assertions are valid: (i) a solution $\{z, f\}$ of the inverse problem (13)-(16) exist and is unique, and (ii) with any initial iteration $f_0 \in L_2(0,T)$ the successive approximations

$$f_{n+1} = \tilde{A} f_n \tag{31}$$

converge to f in the $L_2(0,T)$-norm (for $\tilde{A}_n$ see below).

PROOF. (ii) We have occasion to use the nonlinear operator

$$\tilde{A} : L_2(0,T) \longmapsto L_2(0,T)$$

acting in accordance with the rule

$$\tilde{A} f = Af + \frac{\varphi' + \alpha\varphi}{g_1} \tag{32}$$

where the operator A and the function g_1 arise from (18). From (32) it follows that equation (19) can be written as

$$f = \tilde{A} f. \tag{33}$$

This shows that equation (33) is sufficient to show that operator $\tilde{A}$ has a fixed point in the space $L_2(0,T)$. By the relations

$$\tilde{A}^s f_1 - \tilde{A}^s f_2 = A^s f_1 - A^s f_2 = A^s (f_1 - f_2)$$

we deduce from estimate (29) that

$$\|\tilde{A}^{s_0} f_1 - \tilde{A}^{s_0} f_2\|_{2,(0,T)} = \|A^{s_0}(f_1 - f_2)\|_{2,(0,T)} \leq \frac{K_w^{s_0} \eta^{s_0/2}}{g_0^{s_0} \gamma^{s_0/2}} \|(f_1 - f_2)\|_{2,(0,T)} \tag{34}$$

where s_0 has been fixed in (30). From (30) and (34), we find that $\tilde{A}^{s_0}$ is a contracting mapping on $L_2(0,T)$. Therefore $\tilde{A}^{s_0}$ has a unique fixed point f in $L_2(0,T)$ and the successive approximations (31) converge to f in the $L_2(0,T)$-norm without concern on how the initial iteration $f_0 \in L_2(0,T)$ are chosen.

(i) This shows that, equation (33) and, in turn, equation (19) have a unique solution f in $L_2(0,T)$. According to Theorem 1, this confirms the existence of solution to the inverse problem (13)-(16). It remains to prove the uniqueness of this solution. Assume to the contrary that there were two distinct solutions $\{z_1, f_1\}$ and $\{z_2, f_2\}$ of the inverse problem under consideration. We claim that in that case $f_1 \neq f_2$ almost everywhere on $(0,T)$. If $f_1 = f_2$, then applying the uniqueness theorem to the corresponding direct problem (12)-(14) we would have $z_1 = z_2$ almost everywhere in Q_T. Since both pairs satisfy identity (21), the functions f_1 and f_2 give two distinct solutions to equation (33). But this contradicts the uniqueness of the solution to equation (33) just established and proves the theorem.

COROLLARY 1. Under the conditions of Theorem 2, a solution to equation (19) can be expanded in a series

$$f = \psi + \sum_{s=1}^{\infty} A^s \psi \tag{35}$$

and the estimate

$$\|f\|_{2,(0,T)} \leq \rho \|\psi\|_{2,(0,T)}$$

is valid with

$$\psi = \frac{\varphi' + \alpha\varphi}{g_1}$$

and

$$\rho = \sum_{s=1}^{\infty} \frac{K_w^s \eta^{s/2}}{g_0^s \gamma^{s/2}}.$$

PROOF. The successive approximations (31) with $f_0 = \psi$ verify that

$$f_{n+1} = \tilde{A} f_n = \tilde{A}^n f_0 = \psi + \sum_{s=1}^{\infty} A^s \psi. \tag{36}$$

The passage to the limit as $n \to \infty$ in (36) leads to (35), since by Theorem 2,

$$\|f - f_n\|_{2,(0,T)} \to 0 \quad \text{as} \quad n \to \infty.$$

being concerned with A^s satisfying (29), we get the estimate

$$\|f\|_{2,(0,T)} \leq \|\psi\|_{2,(0,T)} \sum_{s=0}^{\infty} \left(\frac{K_w^{2s} \eta^s}{g_0^{2s} \gamma^s} \right)^{1/2}.$$

By D'Alembert ratio test the series on the right-hand side converges, thereby completing the proof of the theorem.

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