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A Class Of Generalized Filled Functions For Unconstrained Global Optimization*

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Abstract

A new definition of the filled function is given in this paper, which is different from that of the traditional filled functions. A class of generalized filled functions satisfying this definition is presented. We discuss the properties of the proposed filled functions and point out the filled functions previously reported are the special forms of this generalized filled functions. An algorithm based on this new filled functions has been developed. The computational results show that this algorithm is efficient and reliable.

1 Introduction

The filled function method was introduced in Ge's paper [1] for continuous global optimization problem, and many other filled functions have been put forward afterwards in papers [2-8]. The key idea of the filled function method is to leave from a current local minimizer x_1^* to a lower minimizer of the original objective function $f(x)$ with the auxiliary function $P(x)$ constructed at the local minimizer x_1^*. If x_1^* is not a global minimizer of $f(x)$, then $P(x)$ has a local minimizer $\bar{x}$ satisfying $f(\bar{x}) < f(x_1^*)$. With $\bar{x}$ as a new initial point to minimize $f(x)$, we can find a lower minimizer x_2^* of $f(x)$. With x_2^* replacing x_1^*, we can construct a new filled function and then find a much lower minimizer of $f(x)$ in the same way. Repeating the above process, we can finally find the global minimizer x_g^* of $f(x)$.

This paper gives a new definition of the filled function and presents a class of generalized filled functions. It shows that the filled functions given in paper [6] are the special forms of this generalized filled functions.

The rest of the paper is organized as follows: In section 2, some assumptions and a new definition of the filled function are presented. In section 3, a class of generalized filled functions is proposed and its properties are investigated. Two remarks are drawn in section 4. Next, in Section 5, an algorithm and results of numerical experiments for testing several functions are reported. Finally, conclusions are included in Section 6.

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2 New Definition of the Filled Function

We consider the following unconstrained programming problem

$$(P) \quad \min f(x), \quad s.t.\ x \in R^n. \tag{1}$$

We assume that the following conditions are satisfied.

ASSUMPTION 1. $f(x)$ is continuously differentiable on R^n.

ASSUMPTION 2. The minimum of $f(x)$ is a finite set.

ASSUMPTION 3. $f(x)$ is coercive, i.e., $f(x) \to +\infty$, as $\| x \| \to +\infty$.

Notice that Assumption 3 implies the existence of a robust compact set $\Omega \subset R^n$ whose interior contains all minimizers of $f(x)$. We assume that the value of $f(x)$ for x on the boundary of Ω is greater than the value of $f(x)$ for any x inside Ω. Then the original problem (P) is equivalent to the following problem

$$\min f(x), \quad s.t.\ x \in \Omega. \tag{2}$$

Throughout the rest of the paper, let x_1^* be the current local minimizer of $f(x)$, L be the set which consists of all the local minimizers lower than x_1^* of $f(x)$.

DEFINITION 1. A differentiable function $P(x, x_1^*)$ is called a filled function of $f(x)$ at local minimizer x_1^* if $P(x, x_1^*)$ has the following properties:

(i) x_1^* is a strictly local maximizer of $P(x, x_1^*)$;

(ii) If $x \in \Omega \backslash x_1^*$ satisfying $f(x) \geq f(x_1^*)$, then x is not a stationary point of $P(x, x_1^*)$ and satisfies $\nabla^T P(x, x_1^*)(x - x_1^*) < 0$;

(iii) If x_1^* is not a global minimizer of $f(x)$, then $P(x, x_1^*)$ has a minimizer $\bar{x}$ satisfying $f(\bar{x}) < f(x_1^*)$;

(iv) If $x_1,\ x_2 \in \Omega$ satisfying $f(x_1) \geq f(x_1^*)$ and $f(x_2) \geq f(x_1^*)$, then $\|x_2 - x_1^*\| > \|x_1 - x_1^*\|$ if and only if $P(x_2, x_1^*) < P(x_1, x_1^*)$.

Definition 1 shows that x_1^* is a global minimizer of $f(x)$ if and only if x_1^* is the only stationary point of $P(x, x_1^*)$ and x_1^* is a strictly maximizer of $P(x, x_1^*)$. We also know that if x_1^* is not a global minimizer of $f(x)$, then $P(x, x_1^*)$ has a minimizer $\bar{x}$ satisfying $f(\bar{x}) < f(x_1^*)$, therefore, we can obtain a lower local minimizer of $f(x)$ than x_1^* by minimizing $f(x)$ starting from $\bar{x}$.

3 A Class of Generalized Filled Functions and its Properties

In this section we propose a class of generalized filled functions satisfying the Definition 1 and discuss its properties.

For any $r > 0$ and $c > 0$, let

$$g(t, r) = \begin{cases} c & , \quad t \geq 0 \\ h(t, c) & , \quad -r < t < 0 \\ 0 & , \quad t \leq -r \end{cases} \tag{3}$$

where $h(t,c)$ is twice continuously differentiable on R and satisfies $h(0,c)=c$, $h(-r,c)=0$, in this case, $g(t,r)$ is continuously differentiable on R and

$$g'(t,r)=\begin{cases} 0 & , \quad t\geq 0 \\ h'(t,c) & , \quad -r<t<0 \\ 0 & , \quad t\leq -r \end{cases} \tag{4}$$

We define

$$P(x,x_1^*,r,A)=\exp(-A\|x-x_1^*\|^2)\cdot g(f(x)-f(x_1^*),r), \tag{5}$$

where r and $A>0$ are two parameters. Next we will show that $P(x,x_1^*,r,A)$ is a filled function satisfying Definition 1 under certain conditions of the parameters r and A.

THEOREM 1. For any $r>0$ and $A>0$, x_1^* is a strictly local maximizer of $P(x,x_1^*,r,A)$.

PROOF. Since x_1^* is a local minimizer of $f(x)$, there exists a neighborhood of x_1^*, $N(x_1^*,\delta)=\{x|\ \|x-x_1^*\|<\delta\}$ with $\delta>0$. For any $x\in N(x_1^*,\delta)\setminus x_1^*$, we have $f(x)\geq f(x_1^*)$ and

$$g(f(x)-f(x_1^*),r)=c=g(f(x_1^*)-f(x_1^*),r),$$

$$\exp(-A\|x-x_1^*\|^2)<\exp(-A\|x_1^*-x_1^*\|^2),$$

therefore $P(x_1^*,x_1^*,r,A)>P(x,x_1^*,r,A)$ holds for any $r>0$ and $A>0$, i.e., x_1^* is a strictly local maximizer of $P(x,x_1^*,r,A)$.

THEOREM 2. If $x\in\Omega\setminus x_1^*$ satisfying $f(x)\geq f(x_1^*)$, then x is not a stationary point of $P(x,x_1^*,r,A)$ and $\nabla^T P(x,x_1^*,r,A)(x-x_1^*)<0$.

PROOF. It follows from formulation (5) that

$$\begin{aligned}\nabla P(x,x_1^*,r,A) \quad = \quad & \exp(-A\|x-x_1^*\|^2)\cdot[-2A(x-x_1^*)\cdot g(f(x)-f(x_1^*),r) \\ & +g'(f(x)-f(x_1^*),r)\cdot\nabla f(x)].\end{aligned}$$

For any $x\in\Omega\setminus x_1^*$ satisfying $f(x)\geq f(x_1^*)$ we have

$$\nabla P(x,x_1^*,r,A)=-2cA\cdot\exp(-A\|x-x_1^*\|^2)\cdot(x-x_1^*)\neq 0,$$

i.e., x is not a stationary point of $P(x,x_1^*,r,A)$ and

$$\nabla^T P(x,x_1^*,r,A)(x-x_1^*)=-2cA\cdot\exp(-A\|x-x_1^*\|^2)\cdot\|x-x_1^*\|^2<0.$$

THEOREM 3. If x_1^* is not a global minimizer of $f(x)$, then $P(x,x_1^*,r,A)$ has a local minimizer $\bar{x}$ satisfying $f(\bar{x})<f(x_1^*)$ when $r\leq\frac{\beta_0}{2}$ and $A>0$, where

$$\beta_0=\max_{x\in L}(f(x_1^*)-f(x)).$$

Furthermore, $\bar{x}$ satisfies the inequality $P(\bar{x},x_1^*,r,A)<P(x_1^*,x_1^*,r,A)$.

PROOF. Since x_1^* is not a global minimizer of $f(x)$, $L\neq 0$. It follows from Assumption 2 that there exists a $\bar{x}\in L$ such that $\beta_0=f(x_1^*)-f(\bar{x})$. When $r\leq\frac{\beta_0}{2}$, we have

$$f(\bar{x})-f(x_1^*)=-\beta_0\leq -2r<-r,$$

and

$$P(\bar{x}, x_1^*, r, A) = 0.$$

Since $f(x)$ is a continuously function on R^n, there exists a neighborhood of $\bar{x}$, $N(\bar{x}, \delta)$, for any $x \in N(\bar{x}, \delta)$, we have $f(x) - f(x_1^*) < -r$ and

$$P(x, x_1^*, r, A) = 0.$$

Therefore, $\bar{x}$ is a local minimizer of $P(x, x_1^*, r, A)$ satisfying

$$P(\bar{x}, x_1^*, r, A) = 0 < c = P(x_1^*, x_1^*, r, A).$$

THEOREM 4. If $x_1,\ x_2 \in \Omega$ satisfying $f(x_1) \geq f(x_1^*)$ and $f(x_2) \geq f(x_1^*)$, then $\|x_2 - x_1^*\| > \|x_1 - x_1^*\|$ if and only if $P(x_2, x_1^*, r, A) < P(x_1, x_1^*, r, A)$.

PROOF. It follows from $f(x_1) \geq f(x_1^*)$ and $f(x_2) \geq f(x_1^*)$ that

$$P(x_1, x_1^*, r, A) = c \cdot \exp(-A\|x_1 - x_1^*\|^2),$$

and

$$P(x_2, x_1^*, r, A) = c \cdot \exp(-A\|x_2 - x_1^*\|^2).$$

Therefore, $\|x_2 - x_1^*\| > \|x_1 - x_1^*\|$ if and only if $P(x_2, x_1^*, r, A) < P(x_1, x_1^*, r, A)$.

From Theorems 1-4, we know that when $r \leq \frac{\beta_0}{2}$ and $A > 0$, function $P(x, x_1^*, r, A)$ is a filled function satisfying the new Definition 1.

4 Two Remarks

REMARK 1. If function $h(t, c)$ has the following form

$$h(t, c) = -\frac{2c}{r^3}t^3 - \frac{3c}{r^2}t^2 + c. \tag{6}$$

or general form

$$h(t, c) = -\frac{qc}{r^{2m+1}} \cdot t^{2m+1} - \frac{(q+1)c}{r^{2m}} \cdot t^{2m} + c,\ \ (m > 1, q \in N), \tag{7}$$

from formulation (3) and (5) we can obtain the filled function $P(x, x_1^*, r, A)$ which has forms given in paper [6], i.e., the filled functions given in paper [6] are the special forms of this generalized filled functions.

REMARK 2. We can propose another general form of the filled function satisfying Definition 1, let

$$M(t, r) = \begin{cases} d & , \quad t \geq 0 \\ N(t, d) & , \quad -r < t < 0 \\ t + r & , \quad t \leq -r \end{cases} \tag{8}$$

where $N(t, d)$ is twice continuously differentiable on R, and $N(0, d) = d$, $N(-r, d) = 0$, in this case, $M(t, r)$ is a continuously differentiable on R and the function

$$\begin{aligned} \phi(x, x_1^*, r, A) = \quad & \exp(-A\|x - x_1^*\|^2) \cdot g(f(x) - f(x_1^*), r) \\ & + M(f(x) - f(x_1^*), r) \end{aligned} \tag{9}$$

is also a class of generalized filled functions satisfying Definition 1. When

$$N(t,d) = \frac{r-2}{r^3} \cdot t^3 + \frac{r-3}{r^2} \cdot t^2 + d, \tag{10}$$

the $\phi(x, x_1^*, r, A)$ has another form given in paper [6], the filled function given in paper [6] is also the special form of this generalized filled function by (9).

5 Algorithm and Numerical Results

The theoretical properties of the new filled function $P(x, x_1^*, r, q)$ discussed in the foregoing sections give us an approach for finding a global minimizer of $f(x)$. We present an algorithm similar to paper [8] in the following:

ALGORITHM 1[8]. (Filled function method)

1. Initial Step

Choose $r = \frac{1}{2}$, and $0 < r_0 < 1$ as the tolerance parameter for terminating the algorithm.

Choose direction $e_i, i = 1, 2, \cdots, k_0$ with integer $k_0 > 2n$, where n is the number of variable.

Choose an initial point $x_1^0 \in \Omega$.

Let $k = 1$.

2. Main Step

1^0. Obtain a local minimizer of prime problem (P) by implementing a local downhill search procedure starting from x_k^0. Let x_k^* be the local minimizer obtained. Let $i = 1$ and $r = \frac{1}{2}$.

2^0. If $i \le k_0$ then go to 4^0, otherwise go to 3^0

3^0. If $r \le r_0$ then terminate the iteration, the x_k^* is the global minimizer of problem (P), otherwise, let $r = r/10$, $i = 1$, go to 4^0.

4^0. $\bar{x}_k^* = x_k^* + \delta e_i$ (where δ is a very small positive number), if $f(\bar{x}_k^*) < f(x_k^*)$ then Let $k = k + 1$, $x_k^0 = \bar{x}_k^*$ and go to 1^0; otherwise, go to 5^0.

5^0. Let parameter $A = 1$ and $h(t) = -\frac{t^5}{r^5} - \frac{2t^4}{r^4} + 1$ is different from paper [6] and $y_0 = \bar{x}_k^*$. Turn to inner loop.

3. Inner Loop

1^0. Let $m = 0$

2^0. $y_{m+1} = \varphi(y_m)$, where φ is an iteration function. It denotes a local downhill search method for the problem $\min_{s.t x \in \Omega} P(x, x_k^*, r, A)$.

3^0. If $y_{m+1} \notin \Omega$, then let $i = i + 1$, go to main step 2^0, otherwise go to 4^0.

4^0. If $f(y_{m+1}) \le f(x_k^*)$ then let $k = k + 1$, $x_k^0 = y_{m+1}$ and go to main step 1^0, otherwise let $m = m + 1$ and go to 2^0

Now we apply the above algorithm to several test examples proposed in papers [1, 6, 8]. The proposed algorithm is programmed in Fortran 95 for working on the windows XP system with Intel c1.7G CPU and 256M RAM. Numerical results prove that the method is efficient and better than the one given in paper [6].

(i) 6-hump back camel function in [6]

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4,$$

$$-3 \le x_1, x_2 \le 3.$$

The global minimum solutions: $x^* = (0.0898, 0.7127)$ or $(-0.0898, -0.7127)$ and $f^* = -1.0316$. The time to obtain the global minimizer is 26.3325 seconds. The numbers of the filled function and original objective function being calculated in algorithm are 1142 and 1311 respectively.

(ii) Rastrigin function in [1,6]

$$f(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2),$$

$$-1 \le x_1, x_2 \le 1.$$

The global minimum solution: $x^* = (0.0000, 0.0000)$ and $f^* = -2.0000$. The time to obtain the global minimizer is 33.7326 seconds. The numbers of the filled function and original objective function being calculated in algorithm are 2041 and 1854 respectively.

(iii) 2-dimensional function in [8]

$$f(x) = [1 - 2x_2 + c\sin(4\pi x_2) - x_1]^2 + [x_2 - 0.5\sin(2\pi x_1)]^2,$$

$$-10 \le x_1, x_2 \le 10.$$

where $c = 0.2$. The global minimum solution: $f^* = 0.0000$ for all c. The time to obtain the global minimizer is 38.2165 seconds. The numbers of the filled function and original objective function being calculated in algorithm are 2511 and 1915 respectively.

(iv) n-dimensional Sine-square function I in [6,8]

$$f(x) = \frac{\pi}{n}\left\{10\sin^2(\pi x_1) + \sum_{i=1}^{n-1}[(x_i - 1)^2(1 + 10\sin^2(\pi x_{i+1})] + (x_n - 1)^2\right\},$$

$$-10 \le x_1, x_2 \le 10, \quad i = 1, 2, \ldots, n.$$

The function is tested for $n = 10$. The global minimum solution: $x^* = (1.00, \ldots, 1.00)$ and $f^* = 0.00$. The time to obtain the global minimizer is 57.6214 seconds. The numbers of the filled function and original objective function being calculated in algorithm are 4210 and 2648 respectively.

6 Conclusions

This paper gives a new definition of the filled function which is different from that of the traditional filled functions. A class of generalized filled functions satisfying this new definition is presented. Furthermore, we discuss the properties of the proposed filled functions and point out the filled functions previously reported in paper [6] are the special forms of this generalized filled functions. The results of numerical experiments for testing several functions show that the algorithm given in this paper is efficient and reliable.

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