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Construction Of Binomial Sums For π And Polylogarithmic Constants Inspired By BBP Formulas*

Boris Gourévitch†, Jesús Guillera Goyanes‡

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Abstract

We present new sums involving binomial coefficients for π and various logarithms and polylogarithms constants. These sums are a generalization of BBP formulas first introduced by D. Bailey, P. Borwein and S. Plouffe in 1995. In this paper, we describe how to find and prove such sums using the Beta function at integer and rational arguments.

1 Introduction

In 1995, David Bailey, Peter Borwein and Simon Plouffe discovered a new seminal formula for π

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right). \tag{1}$$

This amazing formula allows extracting the n-th binary digit of π without knowledge of previous digits [1]. Such sums are called BBP (Bailey-Borwein-Plouffe) formulas. Many new BBP formulas for π, G (Catalan's constant) or $\zeta(3)$ (Riemann Zeta function) were subsequently discovered by Adamchik [2], Bellard [3], Broadhurst [4], Lupas [5] and Huvent [6]. The main tool for finding BBP formulas is an algorithm designed to detect linear combinations on the set of natural numbers $\mathbb{N}$ like PSLQ (from its use of a partial-sum-of-squares (PS) vector and lower-diagonal-orthogonal (LQ) matrix factorization) or LLL (Lenstra–Lenstra–Lovász) [7], [8]. To then prove such formulas, an integral is evaluated giving rise to logarithms or polylogarithms ($Li_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}$, [9]).

In October 2001, Almkvist and colleagues presented new sums for π involving binomial coefficients such as

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n}. \tag{2}$$

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†17 rue Montera, 75012 Paris, email: boris@pi314.net

‡Av. Cesareo Alierta, 31 esc. izda 4oVA, Zaragoza

They proved this formula using the so-called Beta method which is based on use of Beta function $B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$ for positive integer values r and s [10].

In this paper, using an adapted Beta method, we show how to prove some more general sums of the form

$$T = \sum_{n=0}^{\infty} \frac{1}{q^n \binom{mn}{pn}} \left(\frac{b_1}{mn+1} + \frac{b_2}{mn+2} + \cdots + \frac{b_k}{mn+k} + \cdots + \frac{b_{m-1}}{mn+(m-1)} \right) \quad (3)$$

where T is a constant often related to π and the natural logarithm $\ln(z)$. These sums involve binomial coefficients and they are similar to BBP formulas. We therefore call them "BBP binomial formulas". One example of such formulas is

$$\pi = \frac{1}{16807} \sum_{n=0}^{\infty} \frac{1}{2^n \binom{7n}{2n}} \left(\frac{59296}{7n+1} - \frac{10326}{7n+2} - \frac{3200}{7n+3} - \frac{1352}{7n+4} - \frac{792}{7n+5} + \frac{552}{7n+6} \right) \quad (4)$$

that we discovered with LLL algorithm. An explicit proof of this formula is detailed in this paper. Then we show how to generate an infinite set of similar formulas for π and the natural logarithm $\ln(z)$. We also present outlines for proofs of formulas showing that Beta method is not restrained to integer parameters.

2 Proofs of BBP Binomial Formulas

Consider the Eq. 3. Proof of such formula is reduced to the following proposition:

PROPOSITION 1 (The Beta Method). There exists a polynomial $P(X) \in \mathbb{Q}_{m-2}[X]$ (polynomial of degree $m-2$ with coefficients in the set of rational numbers $\mathbb{Q}$) such that

$$T = \int_0^1 \frac{qP(x)}{q - x^p(1-x)^{(m-p)}} dx \quad (5)$$

PROOF. The Beta function $B(r,s)$ is

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} \quad (6)$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function. With parameters $m, p, n \in \mathbb{N}$, we use $\Gamma(n) = (n-1)!$ to obtain

$$\frac{1}{\binom{mn}{pn}} = (mn+1) \int_0^1 x^{pn}(1-x)^{(m-p)n}\, dx. \quad (7)$$

Summing Eq. (7) on n with $\frac{1}{(mn+1)\binom{mn}{pn}}$ on the left side, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{q^n \binom{mn}{pn}} \frac{1}{mn+1} &= \sum_{n=0}^{\infty} \frac{1}{q^n} \int_0^1 x^{pn}(1-x)^{(m-p)n}\, dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \left(\frac{x^p(1-x)^{(m-p)}}{q} \right)^n dx \\
&= \int_0^1 \frac{q}{q - x^p(1-x)^{(m-p)}} dx. \qquad (8)
\end{aligned}$$

We can evaluate this integral by partial fraction decomposition. More generally, to obtain the elements $\dfrac{1}{mn+k}$ in Eq. (3) we write

$$\int_0^1 x^{pn+k-1}(1-x)^{(m-p)n}\, dx = \frac{1}{\binom{mn+k}{pn+k-1}} = \frac{1}{\binom{mn}{pn}} \frac{(pn+1)(pn+2)...(pn+k-1)}{(mn+1)(mn+2)...(mn+k)}. \quad (9)$$

For a given k, the partial fraction decomposition of $\frac{(pn+1)(pn+2)...(pn+k-1)}{(mn+1)(mn+2)...(mn+k)}$ in n gives $(a_{k,j})_{j=1,..,k}$ such that

$$\int_0^1 x^{k-1} x^{pn}(1-x)^{(m-p)n}\, dx = \frac{1}{\binom{mn}{pn}} \left(\frac{a_{k,1}}{mn+1} + \frac{a_{k,2}}{mn+2} + ... + \frac{a_{k,k}}{mn+k} \right) \quad (10)$$

with $a_{k,j} \in \mathbb{Q}$. For values k ranging from 1 to $m-1$, we linearly combine $(a_{k,j})_{j=1,..,k}$ in order to find the $m-1$ values b_k of Eq. (3). This is equivalent to solving the system

$$\begin{pmatrix} a_{1,1} & a_{2,1} & ... & a_{m-1,1} \\ 0 & a_{2,2} & ... & a_{m-1,2} \\ .... & 0 & ... & ... \\ 0 & 0 & ... & a_{m-1,k} \end{pmatrix} V = \begin{pmatrix} b_1 \\ b_2 \\ .... \\ b_{m-1} \end{pmatrix}. \quad (11)$$

Vector V contains the coefficients of the polynomial $P(X) = V^t . \begin{pmatrix} 1 \\ X \\ .... \\ X^{m-2} \end{pmatrix}$ of degree $m-2$, such that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{q^n \binom{mn}{pn}} \left(\frac{b_1}{mn+1} + \frac{b_2}{mn+2} + \cdots + \frac{b_{m-1}}{mn+(m-1)} \right) \\
&= \sum_{n=0}^{\infty} \int_0^1 P(x) \frac{x^{pn}(1-x)^{(m-p)n}}{q^n}\, dx \\
&= \int_0^1 \frac{qP(x)}{q - x^p(1-x)^{(m-p)}} dx
\end{aligned}$$

when we use Eq. (10).

REMARK 1. Almkvist et al. [10] use the relation $\frac{1}{\binom{3n}{n}} = (3n+1)\int_0^1 x^{2n}(1-x)^n dx$ in order to prove the formula in Eq. (2). In general, they consider formulas of the form

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{mn}{pn} q^n} \tag{12}$$

where $S \in \mathbb{Q}_{m-2}[X]$ and $q \in \mathbb{N}$. However, such forms of sums lead to the intricate evaluation of $f(x) = \sum_{n=0}^{\infty} (mn+1)S(n)\left(\frac{x^p(1-x)^{m-p}}{q}\right)^n$. $f(x)$ is indeed an integral whose denominator is a power of the one considered in Eq. (8). Our method uses $\frac{1}{(mn+1)\binom{mn}{pn}}$ on the left side of Eq. (7), which produces formulas of the general form of Eq. 3.

3 Proof of Eq. 4

We write Eq. (4) as

$$\sum_{n=0}^{\infty} U_n = \pi$$

Using Proposition 1, we are looking for the polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty} U_n = \int_0^1 P(x) \sum_{n=0}^{\infty} \frac{x^{2n}(1-x)^{5n}}{2^n}\, dx = \int_0^1 P(x) \frac{1}{1 - \dfrac{x^2(1-x)^5}{2}} dx$$

$$= \int_0^1 P(x) \frac{2}{(x^2-2x+2)(x^5-3x^4+2x^3+x+1)} dx.$$

Because $\int_0^1 \dfrac{4}{(x^2-2x+2)} dx = \pi$, it is likely that $P(x) = 2\left(x^5 - 3x^4 + 2x^3 + x + 1\right)$. Let us verify this conjecture.

To obtain the equivalent linear system in (11), and therefore vector V, we calculate all coefficients $(a_{k,i})_i$ (columns of left matrix in Eq. (11)) with help from the computation of the integral in Eq. (10). For example, for $k = 2$ (2nd column of matrix), we have

$$\int_0^1 x.x^{2n}(1-x)^{5n}\, dx = B(5n+1, 2n+2) = \frac{1}{\binom{7n}{2n}} \frac{(2n+1)}{(7n+1)(7n+2)}$$

$$= \frac{1}{\binom{7n}{2n}} \left(\frac{5}{7} \frac{1}{(7n+1)} - \frac{3}{7} \frac{1}{(7n+2)} \right).$$

Consequently, $a_{2,1} = \frac{5}{7}$ and $a_{2,2} = -\frac{3}{7}$. The whole linear system is

$$\begin{pmatrix} 1 & \frac{5}{7} & \frac{30}{49} & \frac{190}{343} & \frac{1235}{2401} & \frac{8151}{16807} \\ 0 & -\frac{3}{7} & -\frac{30}{49} & -\frac{255}{343} & -\frac{2040}{2401} & -\frac{15810}{16807} \\ 0 & 0 & \frac{4}{49} & \frac{60}{343} & \frac{660}{2401} & \frac{6380}{16807} \\ 0 & 0 & 0 & \frac{13}{343} & \frac{260}{2401} & \frac{3510}{16807} \\ 0 & 0 & 0 & 0 & -\frac{99}{2401} & -\frac{2475}{16807} \\ 0 & 0 & 0 & 0 & 0 & \frac{276}{16807} \end{pmatrix} V = \begin{pmatrix} \frac{59296}{16807} \\ -\frac{10326}{16807} \\ -\frac{3200}{16807} \\ -\frac{1352}{16807} \\ -\frac{792}{16807} \\ \frac{552}{16807} \end{pmatrix}.$$

Its solution is $V = \begin{pmatrix} 1 & 1 & 0 & 2 & -3 & 1 \end{pmatrix}$. Finally, we have

$$P(X) = V^t \begin{pmatrix} 1 \\ X \\ .... \\ X^5 \end{pmatrix} = 2\left(1 + X + 2X^3 - 3X^4 + X^5\right)$$

which is the expected polynomial.

4 Construction and Prediction of BBP Binomial Formulas

Using proposition 1, we choose a suitable $P(x)$ that simplifies $\frac{qP(x)}{q - x^{(m-p)}(1-x)^p}$. Knowing P then gives rise to a BBP formula by using Eq. (10), after combination and simplification.

As an example, we find formulas for π choosing $P(X)$ such that $\frac{qP(x)}{q - x^{(m-p)}(1-x)^p} = \frac{1}{1+x^2}$ due to $\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$. To find m and p, we have to know when $1 + X^2$ divides $q - X^{(m-p)}(1-X)^p$ *i.e.* when i and $-i$ are roots of $q - X^{(m-p)}(1-X)^p$. Case $m = p$ in Eq. (3) is excluded since it only provides classical BBP formulas without binomial coefficients.

Let us choose $p = 2$, $m = 3$, then $i(1-i)^2 = i(1+i)^2 = 2$. Now there exists a formula for π with $\binom{3n}{2n}$, $p = 2$ and $q = 2$. Moreover, i and $-i$ are fourth roots of 1, so we can multiply $x(1-x)^2$ by x^{4k} and keep roots i and $-i$. Thus there exists formulas for π involving $C^{2n}_{(3+4k)n}$, $k \in \mathbb{N}$.

More generally, if $K(X) = q - X^{(m-p)}(1-X)^p$,

$\{i \text{ and } -i \text{ are roots of } K,\ p < m\} \Leftrightarrow \left\{q - (\pm i)^{(m-p)}(1 \mp i)^p = 0, p < m\right\}$

$\Leftrightarrow \{(1 \pm i)^p = (-1)^{m+p} q(\pm i)^m, p < m\}$

$\Leftrightarrow \{(1 + i)^p = (-1)^{m+p} q i^m, p < m\}$ by complex conjugate

$\Leftrightarrow \left\{\left(\sqrt{2}\right)^p = q \text{ and } p\frac{\pi}{4} \equiv (m+p)\pi + m\frac{\pi}{2}\ [2\pi], p < m\right\}$ with modulus and argument, where $[x]$ designates modulo x

$\Leftrightarrow \left\{2^{\frac{p}{2}} = q,\ p < m \text{ and } -p \equiv 2m\,[8]\right\}$

$\Leftrightarrow \left\{p = 2k, k \in \mathbb{N},\ 2^k = q,\ p < m \text{ and } 7p \equiv 2m\,[8]\right\}$.

This idea also leads to other constants using the root -1 ($\ln(2) = \int_0^1 \frac{1}{1+x}dx$), or the polynomial $x^2 - x + 1$ ($\pi\sqrt{3} = \frac{9}{2}\int_0^1 \frac{1}{1-x+x^2}dx$).

For non-alternating sums $\sum_{n=0}^{\infty} \frac{1}{q^n \binom{mn}{pn}} \left(\frac{b_1}{mn+1} + \frac{b_2}{mn+2} + ... + \frac{b_{m-1}}{mn+(m-1)}\right)$ with equivalent integral representation $\int_0^1 \frac{qP(x)}{q - x^p(1-x)^{(m-p)}}dx$, we obtain the following table for

$p < m$:

Polynomial	Constant	Condition given by roots	Existence of a sum for $p < m$
$1+x^2$	π	$(i)^{(m-p)}(1-i)^p = q$	$p=2k,\ 2^k=q, 7p\equiv 2m\,[8]$
$1+x$	$\ln(2)$	$(-1)^{m-p}2^p = q$	$m=p+2k,\ q=2^p$
$1-x+x^2$	$\pi\sqrt{3}$	$(-1)^p\left(\frac{1+i\sqrt{3}}{2}\right)^{p+m} = q$	$q=1,\ m=2p+6k$
$2+x$	$\ln\left(\frac{3}{2}\right)$	$(-2)^{(m-p)}(3)^p = q$	$m=p+2k,\ q=2^{2k}3^p$
$z-1+x$	$\ln\left(\frac{z+1}{z}\right)$	$(1-z)^{(m-p)}(z)^p = q$	$m=p+2k,\ q=(1-z)^{2k}z^p$
$1+(z-1)x$	$\ln(z)$	$\left(-\frac{1}{z-1}\right)^{(m-p)}\left(1+\frac{1}{z-1}\right)^p = q$	$m=p+2k,\ q=\frac{z^p}{(z-1)^{p+2k}}$

For alternating sums $\sum_{n=0}^{\infty}\frac{(-1)^m}{q^n\binom{mn}{pn}}\left(\frac{b_1}{mn+1}+\frac{b_2}{mn+2}+...+\frac{b_{m-1}}{mn+(m-1)}\right)$ with equivalent integral representation $\int_0^1\frac{qP(x)}{q+x^p(1-x)^{(m-p)}}dx$, we get the following table:

Polynomial	Constant	Condition given by roots	Existence of a sum for $p < m$
$1+x^2$	π	$(i)^{(m-p)}(1-i)^p = -q$	$p=2k,\ 2^k=q, 7p\equiv 2m+4\,[8]$
$1+x$	$\ln(2)$	$(-1)^{m-p+1}2^p = q$	$m=p+2k+1,\ q=2^p$
$1-x+x^2$	$\pi\sqrt{3}$	$(-1)^{p+1}\left(\frac{1+i\sqrt{3}}{2}\right)^{p+m} = q$	$q=1,\ m=2p+3+6k$
$2+x$	$\ln\left(\frac{3}{2}\right)$	$(-2)^{(m-p)}(3)^p = -q$	$m=p+2k+1,\ q=2^{2k}3^p$
$z-1+x$	$\ln\left(\frac{z+1}{z}\right)$	$(1-z)^{(m-p)}(z)^p = -q$	$m=p+2k+1,\ q=(1-z)^{2k}z^p$
$1+(z-1)x$	$\ln(z)$	$\left(-\frac{1}{z-1}\right)^{(m-p)}\left(1+\frac{1}{z-1}\right)^p = q$	$m=p+2k+1,\ q=\frac{z^p}{(z-1)^{p+2k}}$

5 Generalized Beta Function Method

Albeit more intricate, some similar formulas can also be found using fractional parameters in Beta function. Considering Beta function and $\frac{a}{b},\frac{c}{d}\in\mathbb{Q}$, we have

$$\begin{aligned}\int_0^1 x^{pn+\frac{a}{b}}(1-x)^{mn+\frac{c}{d}}dx &= \frac{\Gamma\left(pn+\frac{a}{b}+1\right)\Gamma\left(mn+\frac{c}{d}+1\right)}{\Gamma\left((p+m)n+\frac{a}{b}+\frac{c}{d}+2\right)}\\ &= \frac{\left(\frac{a}{b}\right)_{pn+1}\left(\frac{c}{d}\right)_{mn+1}\Gamma\left(\frac{a}{b}\right)\Gamma\left(\frac{c}{d}\right)}{\left(\frac{a}{b}+\frac{c}{d}\right)_{(p+m)n+2}\Gamma\left(\frac{a}{b}+\frac{c}{d}\right)}\end{aligned} \tag{13}$$

where $(x)_n=\dfrac{\Gamma(x+n)}{\Gamma(x)}=x(x+1)\cdots(x+n-1)$ is Pochhammer symbol. $\Gamma\left(\frac{a}{b}\right)$ is often known for small integers values of a et b. All following formulas stem from summing this relation on n for particular values of $\frac{a}{b},\frac{c}{d}$. We now emphasize three particular cases of interest.

5.1 Case 1 : $\frac{a}{b} = -\frac{1}{2}$, $\frac{c}{d} = 0$

We have

$$\int_0^1 \frac{1}{\sqrt{x}} \frac{q}{q - x^p(1-x)^m} dx = \int_0^1 \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{x^{pn}(1-x)^{mn}}{q^n} dx$$
$$= \sum_{n=0}^{\infty} \int_0^1 q^{-n} x^{pn-\frac{1}{2}} (1-x)^{mn} dx$$
$$= \sum_{n=0}^{\infty} q^{-n} \frac{\Gamma\left(pn + \frac{1}{2}\right)\Gamma(mn+1)}{\Gamma\left((p+m)n + \frac{1}{2} + 1\right)}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2pn)!(mn+pn)!(mn)!}{q^n(2mn+2pn)!(pn)!}. \tag{14}$$

EXAMPLE 1. Case $p = 3$, $m = 1$. We have

$$-\pi = -2\int_0^1 \frac{1}{\sqrt{x}(x+1)} dx = \int_0^1 \frac{1}{\sqrt{x}} \left(x^3 - 2x^2 + 2x - 2\right)\left(\frac{2}{2 - x^3(1-x)}\right) dx$$
$$= \int_0^1 \frac{1}{\sqrt{x}} \left(x^3 - 2x^2 + 2x - 2\right) \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}(1-x)^n}{2^n} dx$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \int_0^1 \left(x^3 - 2x^2 + 2x - 2\right) \frac{x^{3n}}{\sqrt{x}} (1-x)^n dx \tag{15}$$

So now using Eq. (14), we have

$$-\int_0^1 \left(x^3 - 2x^2 + 2x - 2\right) \frac{x^{3n}}{\sqrt{x}} (1-x)^n dx$$
$$= -[-1, 2, -2, 2] \times \int_0^1 x^{3n + \left[\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right]^T} (1-x)^n dx$$
$$= [-1, 2, -2, 2] \times \frac{\Gamma\left(3n + \left[\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right]^T\right)\Gamma(n+1)}{\Gamma\left(4n + \left[\frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}\right]^T\right)}$$
$$= \frac{(6n)!(4n)!n!4^n}{(3n)!(8n)!} \frac{1}{512} \left(\frac{1885}{8n+1} + \frac{-965}{8n+3} + \frac{363}{8n+5} + \frac{-51}{8n+7}\right) \tag{16}$$

in simplified notations. Finally with Eq. (15) and Eq. (16), we have

$$2^9\pi = \sum_{n=0}^{\infty} (-1)^n 2^n \frac{\binom{6n}{3n}}{\binom{8n}{4n}\binom{4n}{n}} \left(\frac{1885}{8n+1} + \frac{-965}{8n+3} + \frac{363}{8n+5} + \frac{-51}{8n+7}\right).$$

5.2 Case 2 : $\frac{a}{b} + \frac{c}{d} \in \mathbb{Z}$ (positive and negative integers)

This case allows us to obtain special values of $\Gamma\left(\frac{a}{b}\right)\Gamma\left(\frac{c}{d}\right)$ (equal to π multiplied by an algebraic number). In some cases, π is absent and we obtain amazing formulas for algebraic numbers.

EXAMPLE 2. The following relations are based on Beta values for $\frac{a}{b}=-\frac{4}{3},\frac{c}{d}=-\frac{2}{3}$ or $\frac{a}{b}=-\frac{1}{3},\frac{c}{d}=-\frac{2}{3}$ or $\frac{a}{b}=-\frac{5}{3},\frac{c}{d}=-\frac{1}{3}$

$$\frac{\binom{3n}{n}}{27^n(3n-1)}=\frac{\sqrt{3}}{4\pi}\int_0^1 x^{n-\frac{4}{3}}(1-x)^{n-\frac{2}{3}}dx-\frac{\sqrt{3}}{2\pi}\int_0^1 x^{n-\frac{1}{3}}(1-x)^{n-\frac{2}{3}}dx$$

$$\frac{\binom{3n}{n}}{27^n(3n-2)}=\frac{\sqrt{3}}{8\pi}\int_0^1 x^{n-\frac{5}{3}}(1-x)^{n-\frac{1}{3}}dx-\frac{\sqrt{3}}{4\pi}\int_0^1 x^{n-\frac{1}{3}}(1-x)^{n-\frac{2}{3}}dx$$

which gives the formula

$$\frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{3n}{n}}{54^n}\left(\frac{-1}{3n-1}+\frac{1}{3n-2}\right)=\sqrt[3]{2}.$$

5.3 Case 3 : $\frac{a}{b}+\frac{c}{d}\in\mathbb{Q}\backslash\mathbb{N}$ (rationals but not naturals)

We obtain Pochhammer symbols in sums then factors $\dfrac{\Gamma\left(\frac{a}{b}\right)\Gamma\left(\frac{c}{d}\right)}{\Gamma\left(\frac{a}{b}+\frac{c}{d}\right)}$ in the outcome.

EXAMPLE 3. Considering

$$\int_0^1 x^{n-\frac{1}{4}}(1-x)^{n-\frac{1}{2}}dx=\frac{\Gamma\left(n+\frac{3}{4}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(2n+\frac{5}{4}\right)}$$

$$\int_0^1 x^{n+\frac{3}{4}}(1-x)^{n-\frac{1}{2}}dx=\frac{\Gamma\left(n+\frac{7}{4}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(2n+\frac{9}{4}\right)}$$

and the following integral

$$\int_0^1(x-2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^{n-\frac{1}{4}}(1-x)^{n-\frac{1}{2}}dx=-2\int_0^1\frac{1}{\sqrt[4]{x}\sqrt{1-x}(x+1)}dx=-\pi$$

we obtain finally

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{2n}}{\left(\frac{1}{4}\right)_n\left(\frac{1}{4}\right)_{2n}}\left(\frac{11}{8n+1}+\frac{1}{8n+5}\right)=\sqrt{2}\frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{\pi}}=4\sqrt{2}K\left(\frac{1}{\sqrt{2}}\right)$$

where $K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is the complete elliptic integral of the first kind, thus making the connection with singular values of the elliptic integral.

6 Higher Order Constants

One point of interest is to generalize this method to higher order constants. A constant is of order p if it is equal to a linear combination in $\mathbb{N}$ of p-th order polylogarithms. For example, π or $\ln(2)$ are of order 1 whereas $\pi\ln(2)$ or G are of order 2 and $\zeta(3)$ or π^3

are of order 3. A BBP binomial sum involves a constant of order p, as for BBP sums, if the polynomial in n in the denominator is of degree p. For example, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n} 2^n \, (n+1)^2} \left(\frac{3}{(2n+1)} \right) = -2 \ln^3(2) + 3\zeta(3). \tag{17}$$

However, the straightforward application of integral representation does not completely help since, for order 2 for instance, we have

$$\sum_{n=0}^{\infty} \frac{1}{q^n \binom{mn}{pn} (mn+1)^2} = \int_0^1 \int_0^1 \frac{q}{q - y^m x^m (1-x)^{(m-p)}} dxdy.$$

The evaluation is not often easy, there is no known proof of Eq. (17) to our knowledge. Nevertheless, there is evidence for existence of BBP binomial formulas for higher order constants.

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