Applied Mathematics E-Notes, 7(2007), 214-221 © ISSN 1607-2510
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Error Analysis Of Adomian Series Solution To A Class Of Nonlinear Differential Equations*

Ibrahim L. El-Kalla†

Received 6 August 2006

Abstract

In this paper, a new formula for Adomian polynomials is introduced. Based on this new formula, error analysis of Adomian series solution for a class of nonlinear differential equations is discussed. Numerical experiment shows that Adomian solution using this new formula converges faster.

1 Introduction

Recently, a great deal of interest has been focused on the convergence studies of the series-solution obtained using Adomian Decomposition Method (ADM) for a wide variety of stochastic and deterministic problems [1-4]. Convergence of ADM when applied to some classes of ordinary differential equations is discussed by many authors for example [5,6]. For linear operator equations, Golberg [7] shows that ADM is equivalent to the classical methods of successive approximation (Picard iteration). Lesnic [8] investigates the convergence of ADM when applied to time-dependent problems governed by the heat, wave and beam equations for both forward and backward problems. It is shown that for forward problems the convergence is faster than for backward problems. An efficient technique based on Adomian method for computing the eigenelements of fourth-order Sturm-Liouville boundary value problems is developed in [9]. Al-Khaled and Allan [10] implemented ADM for variable-depth shallow water equations with source term and the convergence is illustrated numerically. A comparative study between ADM and Sinc-Galerkian method for solving some population growth models is performed by Al-Khaled [11] and between ADM and Runge Kutta method for solving system of ordinary differential equations is performed by Shawagfeh et al. [12]. In these comparisons, it is found that ADM offers a simple and more accurate approximate solution. Further important concrete applications of ADM to different types of functional equations are discussed [13-17]. The contribution of the work reported in this paper can be summarized in the following four points:

- Introducing a new formula for the Adomian polynomials (see section 3)

*Mathematics Subject Classifications: 34L30, 35A35.

†Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, PO 35516 Mansoura, Egypt.

• Introducing the sufficient condition that guarantees existence of a unique solution to the problem (see Theorem 1)

• Based on the above two points, convergence of ADM when applied to the problem is proved (see Theorem 2)

• The maximum absolute error of the Adomian truncated series solution is estimated (see Theorem 3).

2 Standard ADM Applied to the Problem

Consider the k^{th} order nonlinear ordinary differential equation

$$\frac{d^k}{dt^k}\,y(t) + \beta(t) f(y) = x(t), \tag{1}$$

subjected to suitable initial conditions

$$y(0) = c_0, \frac{dy(0)}{dt} = c_1, \frac{d^2y(0)}{dt^2} = c_2, ..., \frac{d^{k-1}y(0)}{dt^{k-1}} = c_{n-1}, \tag{2}$$

where $c_0, c_1, c_2, ..., c_{n-1}$ are finite constants. In this work $x(t)$ is assumed to be bounded $\forall\, t \in J = [0, T]$ and $|\beta(\tau)| \le M \;\forall\; 0 \le \tau \le t \le T$, M is a finite constant. The nonlinear term $f(y)$ is Lipschitzian with $|f\,(y) - f\,(z)| \le L\,|y - z|$ and has Adomian polynomials representation

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, ..., y_n), \tag{3}$$

where the traditional formula of A_n is

$$A_n = (1/n!)(d^n/d\lambda^n)\left[f\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}. \tag{4}$$

Using equation (3) in equation (1) we get

$$\pounds y\,(t) + \beta(t) \sum_{n=0}^{\infty} A_n = x(t) \tag{5}$$

where $\pounds = \frac{d^k}{dt^k}$. Applying $\pounds^{-1}$ on both sides of equation (5) to obtain

$$y\,(t) = \theta\,(t) + \pounds^{-1} x\,(t) - \pounds^{-1}\beta(t) \sum_{n=0}^{\infty} A_n \tag{6}$$

where, $\theta(t)$ is the solution of $\pounds\theta\,(t) = 0$ satisfied by the given initial conditions and $\pounds^{-1}\,(.) = \int_0^t \cdots_{k-fold} \cdots \int_0^t (.)\, dt...dt$. Application of ADM to (6) yields

$$y_0\,(t) = \theta\,(t) + \pounds^{-1} x\,(t)\,, \tag{7}$$

and

$$y_i\left(t\right) = -\pounds^{-1}\beta(\tau)A_{i-1}, \quad i \geq 1. \tag{8}$$

Finally, the Adomian series solution is

$$y(t) = \sum_{i=0}^{\infty} y_i(t). \tag{9}$$

The Adomian's polynomials are not unique and it can be generated from Taylor expansion of $f\left(y\right)$ about the first component y_0 i.e. $f\left(y\right) = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{[y-y_0]^n}{n!} f^{(n)}\left(y_0\right)$ [18, 19]. In [19] Adomian's polynomials are arranged to have the form

$$A_0 = f\left(y_0\right),$$
$$A_1 = y_1 f^{(1)}\left(y_0\right),$$
$$A_2 = y_2 f^{(1)}\left(y_0\right) + \frac{1}{2!} y_1^2 f^{(2)}\left(y_0\right),$$
$$A_3 = y_3 f^{(1)}\left(y_0\right) + y_1 y_2 f^{(2)}\left(y_0\right) + \frac{1}{3!} y_1^3 f^{(3)}\left(y_0\right).$$
$$\vdots$$

3 A New Formula to Adomian's Polynomials

By rearranging the terms in the old polynomials yields a new definition of Adomian's polynomials as follow:

$$\bar{A}_0 = f\left(y_0\right),$$
$$\bar{A}_1 = y_1 f^{(1)}\left(y_0\right) + \frac{1}{2!} y_1^2 f^{(2)}\left(y_0\right) + \frac{1}{3!} y_1^3 f^{(3)}\left(y_0\right) + \ldots$$
$$\bar{A}_2 = y_2 f^{(1)}\left(y_0\right) + \frac{1}{2!}\left(y_2^2 + 2y_1y_2\right) f^{(2)}\left(y_0\right) + \frac{1}{3!}\left(3y_1^2y_2 + 3y_1y_2^2 + y_2^3\right) f^{(3)}\left(y_0\right) + \ldots$$

$$\begin{aligned} \bar{A}_3 &= y_3 f^{(1)}\left(y_0\right) + \frac{1}{2!}\left(y_3^2 + 2y_1y_3 + 2y_2y_3\right) f^{(2)}\left(y_0\right) \\ &\quad + \frac{1}{3!}\left(y_3^3 + 3y_3^2\left(y_1+y_2\right) + 3y_3\left(y_1+y_2\right)^2\right) f^{(3)}\left(y_0\right) + \ldots \end{aligned}$$
$$\vdots$$

Define the partial sum $S_n = \sum_{i=0}^{n} y_i(t)$, from the rearranged polynomials we can write

$$\bar{A}_0 = f\left(y_0\right) = f\left(S_0\right),$$

$$\begin{aligned} \bar{A}_0 + \bar{A}_1 &= f\left(y_0\right) + y_1 f^{(1)}\left(y_0\right) + \frac{1}{2!} y_1^2 f^{(2)}\left(y_0\right) + \frac{1}{3!} y_1^3 f^{(3)}\left(y_0\right) + \ldots \\ &= f\left(y_0 + y_1\right) \\ &= f\left(S_1\right). \end{aligned}$$

Similarly, we can find

$$\bar{A}_0 + \bar{A}_1 + \bar{A}_2 = f\left(y_0 + y_1 + y_2\right) = f\left(S_2\right).$$

By induction

$$\sum_{i=0}^{n} \bar{A}_i(y_0, y_1, ..., y_i) = f\left(S_n\right),$$

from which we have

$$\bar{A}_n = f\left(S_n\right) - \sum_{i=0}^{n-1} \bar{A}_i. \tag{10}$$

Which is the formula.

For example, if $f(y) = y^3$ the first four polynomials using formulas (4) and (10) are computed to be:

Using formula (4):

$$A_0 = y_0^3,$$

$$A_1 = 3y_0^2\, y_1,$$

$$A_2 = 3y_0\, y_1^2 + 3y_0^2\, y_2,$$

$$A_3 = y_1^3 + 6y_0\, y_1\, y_2 + 3y_0^2\, y_3,$$

$$A_4 = 3y_1^2\ y_2 + 3y_0\ y_2^2 + 6y_0\, y_1\, y_3 + 3y_0^2\, y_4.$$

Using formula (10):

$$\bar{A}_0 = y_0^3,$$

$$\bar{A}_1 = 3y_0^2 y_1 + 3y_0 y_1^2 + y_1^3,$$

$$\bar{A}_2 = 3y_0^2 y_2 + 3y_0 y_2^2 + 3y_1^2 y_2 + 3y_1 y_2^2 + 6y_0 y_1 y_2 + y_2^3,$$

$$\bar{A}_3 = 3y_0^2 y_3 + 3y_0 y_3^2 + 3y_1^2 y_3 + 3y_1 y_3^2 + 3y_2^2 y_3 + 3y_2 y_3^2 + 6y_0 y_1 y_3 + 6y_0 y_2 y_3 + 6y_1 y_2 y_3 + y_3^3,$$

$$\begin{aligned}\bar{A}_4 \quad = \quad & 3y_0^2 y_4 + 3y_0 y_4^2 + 3y_1^2 y_4 + 3y_1 y_4^2 + 3y_2^2 y_4 + 3y_2 y_4^2 + 3y_3^2 y_4 + 3y_3 y_4^2 \\ & +6y_0 y_1 y_4 + 6y_0 y_2 y_4 + 6y_0 y_3 y_4 + 6y_1 y_2 y_4 + 6y_1 y_3 y_4 + 6y_2 y_3 y_4 + y_4^3.\end{aligned}$$

Clearly, the first four polynomials computed using the suggested formula (10) include the first four polynomials computed using formula (4) in addition to other terms that should appear in $A_5, A_6, A_7, ...$using formula (4). Thus, the solution that obtained using formula (10) enforces many terms to the calculation processes earlier, yielding a faster convergence.

4 Convergence Analysis

In this section the sufficient condition that guarantees existence of a unique solution is introduced in Theorem 1, convergence of the series solution (9) is proved in Theorem 2 and finally the maximum absolute error of the truncated series (9) is estimated in Theorem 3.

THEOREM 1. Problem (1)-(2) has a unique solution whenever $0 < \alpha < 1$, where, $\alpha = \frac{LMT^k}{k!}$

PROOF. Denoting $E = (C[J], \|.\|)$ the Banach space of all continuous functions on J with the norm $\|y(t))\| = \max_{t\in J} |y(t)|$. Define a mapping $F : E \to E$ where $Fy(t) = \theta(t) + \pounds^{-1}x(t) - \pounds^{-1}\beta(\tau)f(y)$. Let, y and $\overset{*}{y} \in E$ we have

$$\begin{aligned}
\left\|Fy - F\overset{*}{y}\right\| &= \max_{t\in J}\left|\pounds^{-1}\beta(\tau)\left[f(y) - f(\overset{*}{y})\right]\right| \\
&\le \max_{t\in J}\pounds^{-1}\ |\beta(\tau)|\ \left|f(y) - f(\overset{*}{y})\right| \\
&\le LM\max_{t\in J}\left|y - \overset{*}{y}\right|\int_0^t \cdots_{k-fold}\cdots\int_0^t dt...dt \\
&\le \frac{LMT^k}{k!}\max_{t\in J}\left|y - \overset{*}{y}\right| \\
&\le \alpha\left\|y - \overset{*}{y}\right\|.
\end{aligned}$$

Under the condition $0 < \alpha < 1$ the mapping F is contraction therefore, by the Banach fixed-point theorem for contraction, there exist a unique solution to problem (1)-(2) and this completes the proof.

THEOREM 2. The series solution (9) of problem (1)-(2) using ADM converges whenever $0 < \alpha < 1$ and $|y_1| < \infty$.

PROOF. Let, S_n and S_m be arbitrary partial sums with $n \ge m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Banach space E

$$\begin{aligned}
\|S_n - S_m\| &= \max_{t\in J}|S_n - S_m| \\
&= \max_{t\in J}\left|\sum_{i=m+1}^{n} y_i(t)\right| \\
&= \max_{t\in J}\left|\sum_{i=m+1}^{n} -\pounds^{-1}\beta(\tau)\bar{A}_{i-1}\right| \\
&= \max_{t\in J}\left|\pounds^{-1}\beta(\tau)\sum_{i=m}^{n-1}\bar{A}_i\, d\tau\right|.
\end{aligned}$$

From (10) we have $\sum_{i=m}^{n-1} \bar{A}_i = f(S_{n-1}) - f(S_{m-1})$ so

$$\begin{aligned} \|S_n - S_m\| &= \max_{t\in J} \left| \pounds^{-1}\beta(\tau)\left[f(S_{n-1}) - f(S_{m-1})\right]\right| \\ &\leq \max_{t\in J} \pounds^{-1}\left|\beta(\tau)\right| \ \left|f(S_{n-1}) - f(S_{m-1})\right| \\ &\leq \frac{LMT^k}{k!}\ \|S_{n-1} - S_{m-1}\| \\ &\leq \alpha\ \|S_{n-1} - S_{m-1}\|. \end{aligned}$$

Let, $n = m + 1$ then

$$\|S_{m+1} - S_m\| \leq \alpha \|S_m - S_{m-1}\| \leq \alpha^2 \|S_{m-1} - S_{m-2}\| \leq ... \leq \alpha^m \|S_1 - S_0\|.$$

From the triangle inequality

$$\begin{aligned} \|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\ &\leq \left[\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}\right]\ \|S_1 - S_0\| \\ &\leq \alpha^m\left[1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}\right]\ \|S_1 - S_0\| \\ &\leq \alpha^m\left(\frac{1-\alpha^{n-m}}{1-\alpha}\right)\ \|y_1(t)\|. \end{aligned}$$

Since $0 < \alpha < 1$ so, $(1 - \alpha^{n-m}) < 1$ then we have

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_1(t)|. \tag{11}$$

But $|y_1| < \infty$ (since $x(t)$ is bounded) so, as $m \to \infty$ then $\|S_n - S_m\| \to 0$. We conclude that $\{S_n\}$ is a Cauchy sequence in E so, the series $\sum_{n=0}^{\infty} y_n(t)$ converges and the proof is complete.

THEOREM 3. The maximum absolute truncation error of the series solution (9) to problem (1)-(2) is estimated to be: $\max_{t\in J}|y(t) - \sum_{i=0}^{m} y_i(t)| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$.

PROOF. From (11) in Theorem 2 we have

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_1(t)|.$$

As $n \to \infty$ then $S_n \to y(t)$ so we have

$$\|y(t) - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_1(t)|,$$

and the maximum absolute truncation error in the interval J is estimated to be

$$\max_{t\in J}\left|y(t) - \sum_{i=0}^{m} y_i(t)\right| \leq \max_{t\in J} \frac{\alpha^m}{1-\alpha}|y_1(t)|. \tag{12}$$

This completes the proof.

4.1 A Numerical Experiment

Consider the nonlinear ordinary differential equation

$$\frac{d^2y}{dt^2} + e^{-2t}y^3 = 2e^t,$$

subject to the initial conditions,

$$y(0) = y'(0) = 1,$$

which has exact solution $y(t) = e^t$. Using MATHEMATICA, this example is solved using new and old polynomials. A comparative study, in table 1 using 7 terms approximation, shows that the solution using the new polynomials (10) converges faster than the solution using the old polynomials (4).

Table (1): Relative Absolute Error (RAE)

t	RAE using old polynomials	RAE using new polynomials
0.1	8.18149×10^{-14}	2.59738×10^{-16}
0.2	6.69176×10^{-12}	7.07935×10^{-15}
0.3	1.71173×10^{-10}	5.14516×10^{-14}
0.4	9.88231×10^{-8}	9.81922×10^{-13}
0.5	4.60176×10^{-6}	1.01016×10^{-11}
0.6	0.0000195279	9.91096×10^{-9}
0.7	0.0000877121	1.06216×10^{-8}
0.8	0.000210942	6.63176×10^{-7}
0.9	0.000785103	3.69176×10^{-6}
1	0.00131653	0.0000110157

5 Conclusion

A new formula for Adomian polynomials is introduced. Based on this new formula, the contraction mapping principles can be employed successfully to estimate the maximum absolute truncated error. Numerical experiment shows that the Adomian series solution using this new formula converges faster.

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