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Boundary Value Problems For A Class Of Singular Second Order Differential Equation*

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Abstract

This paper is concerned with positive solutions to BVPs for a class of singular second order differential equation. By the classical method of elliptic regularization, we prove the existence of positive solutions and generalize a recent work.

1 Introduction

This paper is concerned with the existence of positive solutions for a class of singular second order differential equation

$$\varphi'' - \lambda \frac{|\varphi'|^2}{\varphi} + f(t)(1 + |\varphi'|^2)^m = 0, \quad 0 < t < 1, \tag{1}$$

with one of the following boundary conditions

$$\varphi(1) = \varphi(0) = 0, \tag{2}$$

$$\varphi(1) = \varphi(0) = \varphi'(1) = \varphi'(0) = 0, \tag{3}$$

where $\lambda > 0, m \in (-\infty, 0) \cup (0, \frac{1}{2}], f(t) \in C^1[0,1]$ and $f(t) > 0$ on $[0,1]$.

It is well known that boundary value problems (BVPs) for singular second order ordinary differential equations arise in the fields of gas dynamics, flow mechanics, theory of boundary layer, and so on. In recent years, singular second order ordinary differential equations with dependence on the first order derivative have been studied extensively, see for example [1-8] and references therein where some general existence results were obtained. We point out that the equation considered here is not in their considerations since it does not satisfy some sufficient conditions of those papers. Our considerations were motivated by [9] in which the authors studied (1) with $m = 0, f \equiv 1$ and the boundary conditions: $\varphi(1) = \varphi'(0) = 0$, and, by ordinary differential equation theories, obtained a decreasing positive solution. Recently, in [10] the authors studied (1) with $m = 0$ and the boundary conditions (2) or (3), and, by the classical method of elliptic

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regularization, obtained a positive solution which is not decreasing. In the present paper, we consider (1) and generalize the existence results of [10].

We say $\varphi \in C^2(0,1)\cap C[0,1]$ is a solution to BVP (1),(2) if it is positive in $(0,1)$ and satisfies (1) and (2). Similarly, we say $\varphi \in C^2(0,1)\cap C^1[0,1]$ is a solution to BVP (1),(3) if it is positive in $(0,1)$ and satisfies (1) and (3).

Our main results are the following

THEOREM 1. Let $\lambda > 0$, $f(t)\in C^1[0,1]$ and $f>0$ on $[0,1]$. If $m\in(-\infty,0)\cup(0,\frac{1}{2})$ or $m=\frac{1}{2}$ and $\max_{[0,1]} f<1$, then BVP (1),(2) admits at least a solution.

THEOREM 2. Let $\lambda > \frac{1}{2}$, $f(t)\in C^1[0,1]$ and $f>0$ on $[0,1]$. If $m\in(-\infty,0)\cup(0,\frac{1}{2})$ or $m=\frac{1}{2}$ and $\max_{[0,1]} f<1$, then BVP (1),(3) admits at least a solution.

2 Proofs of Theorem 1 and Theorem 2

We will use the classical method of elliptic regularization to prove Theorem 1. For this, we consider the following regularized problem:

$$\varphi'' - \lambda\frac{|\varphi'|^2}{I_\varepsilon(\varphi)}\mathrm{sgn}_\varepsilon(\varphi) + f(t)(1+|\varphi'|^2)^m = 0, \quad 0<t<1,$$

$$\varphi(1)=\varphi(0)=\varepsilon,$$

where $\varepsilon\in(0,1)$, $I_\varepsilon(s)$ and $\mathrm{sgn}_\varepsilon(s)$ can be defined as follows: $I_\varepsilon(s)=s$ if $s\ge\varepsilon$, $I_\varepsilon(s)=\frac{s^2+\varepsilon^2}{2\varepsilon}$ if $-\varepsilon<s<\varepsilon$ and $I_\varepsilon(s)=-s$ if $s\le-\varepsilon$, and $\mathrm{sgn}_\varepsilon(s)=1$ if $s\ge\varepsilon$, $\mathrm{sgn}_\varepsilon(s)=\frac{2s}{\varepsilon}-\frac{s^2}{\varepsilon^2}$ if $0\le s<\varepsilon$, $\mathrm{sgn}_\varepsilon(s)=\frac{2s}{\varepsilon}+\frac{s^2}{\varepsilon^2}$ if $-\varepsilon\le s<0$ and $\mathrm{sgn}_\varepsilon(s)=-1$ if $s<-\varepsilon$. Clearly, $I_\varepsilon(s),\mathrm{sgn}_\varepsilon(s)\in C^1(\mathbb{R})$, and $I_\varepsilon(s)\ge\varepsilon/2, 1\ge|\mathrm{sgn}_\varepsilon(s)|, \mathrm{sgn}_\varepsilon(s)\mathrm{sgn}(s)\ge 0$ in $\mathbb{R}$.

For $m\le\frac{1}{2}$ and $\lambda>0$, it follows from Theorem 4.1 of Chapter 7 in [11] that for any fixed $\varepsilon\in(0,1)$, the above regularized problem admits a unique classical solution $\varphi_\varepsilon\in C^2(0,1)\cap C^1[0,1]$. By the maximal principle, it is easy to see that $\varphi_\varepsilon(t)\ge\varepsilon$ on $[0,1]$. Thus φ_ε satisfies

$$\varphi_\varepsilon'' - \lambda\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon} + f(t)(1+|\varphi_\varepsilon'|^2)^m = 0, \quad 0<t<1, \tag{4}$$

$$\varphi_\varepsilon(0)=\varphi_\varepsilon(1)=\varepsilon.$$

Note that (4) is equivalent to

$$\left(\int_0^{\varphi_\varepsilon'}\frac{1}{(1+s^2)^m}ds\right)' - \lambda\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon(1+|\varphi_\varepsilon'|^2)^m} + f(t) = 0, \quad 0<t<1, \tag{5}$$

LEMMA 1. Under the assumptions of Theorem 1, for all $\varepsilon\in(0,1)$ there exists a positive constant D independent of ε such that

$$|\varphi_\varepsilon'(t)|\le D, \quad t\in[0,1].$$

PROOF. Noticing $\varphi_\varepsilon(1) = \varphi_\varepsilon(0) = \varepsilon$ and $\varphi_\varepsilon(t) \geq \varepsilon$ for all $t \in [0,1]$, we have

$$\varphi'_\varepsilon(0) = \lim_{t\to 0^+} \frac{\varphi_\varepsilon(t)-\varepsilon}{t} \geq 0,$$

$$\varphi'_\varepsilon(1) = \lim_{t\to 1^-} \frac{\varphi_\varepsilon(t)-\varepsilon}{t-1} \leq 0.$$

On the other hand, it follows from (5) that

$$\left(\int_0^{\varphi'_\varepsilon} \frac{1}{(1+s^2)^m} ds\right)' + A \geq 0, \quad 0 < t < 1,$$

where $A \triangleq \max_{[0,1]} f$, i.e.

$$\left(\int_0^{\varphi'_\varepsilon} \frac{1}{(1+s^2)^m} ds + At\right)' \geq 0, \quad 0 < t < 1.$$

Thus the function $\int_0^{\varphi'_\varepsilon} \frac{1}{(1+s^2)^m} ds + At$ is non-decreasing on $[0,1]$, therefore

$$\begin{aligned} A &\geq \int_0^{\varphi'_\varepsilon(1)} \frac{1}{(1+s^2)^m} ds + A \\ &\geq \int_0^{\varphi'_\varepsilon(t)} \frac{1}{(1+s^2)^m} ds + At \\ &\geq \int_0^{\varphi'_\varepsilon(0)} \frac{1}{(1+s^2)^m} ds \geq 0, \quad t \in [0,1], \end{aligned}$$

and hence

$$\left|\int_0^{\varphi'_\varepsilon(t)} \frac{1}{(1+s^2)^m} ds\right| \leq A, \quad t \in [0,1].$$

Case 1. $m < 0$. In the case, using the inequality $|\int_0^z \frac{1}{(1+s^2)^k} ds| \geq |z|, k < 0, z \in \mathbb{R}$, we obtain

$$|\varphi'_\varepsilon(t)| \leq A, \quad t \in [0,1].$$

Case 2. $m \in (0, \frac{1}{2}]$. Using the inequality

$$\left|\frac{z}{(1+z^2)^k}\right| \leq \left|\int_0^z \frac{1}{(1+s^2)^k} ds\right|, \quad k > 0, z \in \mathbb{R},$$

we obtain

$$\left|\frac{\varphi'_\varepsilon(t)}{(1+|\varphi'_\varepsilon(t)|^2)^m}\right| \leq A, \quad t \in [0,1],$$

i.e.

$$|\varphi'_\varepsilon(t)| \leq A(1+|\varphi'_\varepsilon(t)|^2)^m, \quad t \in [0,1]. \tag{6}$$

If $m \in (0, \frac{1}{2})$, by the inequality $(a+b)^r \le a^r + b^r (a, b \ge 0, r \in [0,1])$ and Young's inequality, we have

$$(1+|\varphi_\varepsilon'|^2)^m \le |\varphi_\varepsilon'|^{2m} + 1 \le \frac{1}{2A}|\varphi_\varepsilon'| + C, \quad t \in [0,1],$$

where C is a positive constant independent of ε, and hence it follows from (6) that

$$|\varphi_\varepsilon'| \le A(1+|\varphi_\varepsilon'|^2)^m \le \frac{1}{2}|\varphi_\varepsilon'| + C, \quad t \in [0,1],$$

therefore

$$|\varphi_\varepsilon'| \le C, \quad t \in [0,1].$$

If $m = \frac{1}{2}$, we have $A = \max_{[0,1]} f < 1$. Then it follows from (6) that

$$|\varphi_\varepsilon'| \le \frac{A}{\sqrt{1-A^2}}, \quad t \in [0,1].$$

This completes the proof of Lemma 1.

Denote D_m and d_m by

$$D_m = \max\{1, (1+D^2)^m\}, \qquad d_m = \min\{1, (1+D^2)^m\},$$

where D is the same as that of Lemma 1. From (4) and Lemma 1 we obtain

$$-\varphi_\varepsilon'' + \lambda\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon} - d_m \min_{[0,1]} f \ge 0, \quad t \in (0,1). \tag{7}$$

$$-\varphi_\varepsilon'' + \lambda\frac{|\varphi_\varepsilon'|^2}{\varphi_\varepsilon} - D_m \max_{[0,1]} f \le 0, \quad t \in (0,1). \tag{8}$$

To obtain the uniform bounds of φ_ε, we need to establish the following comparison theorem.

PROPOSITION 2. Let $\varphi_i \in C^2(0,1) \cap C[0,1]$ and $\varphi_i > 0$ on $[0,1] (i = 1,2)$. If $\varphi_2 \ge \varphi_1$ for $t = 0, 1$, and

$$-\varphi_2'' + \varrho\frac{|\varphi_2'|^2}{\varphi_2} - \theta \ge 0, \quad t \in (0,1), \tag{9}$$

$$-\varphi_1'' + \varrho\frac{|\varphi_1'|^2}{\varphi_1} - \theta \le 0, \quad t \in (0,1), \tag{10}$$

where ϱ and θ are positive constants, then

$$\varphi_2(t) \ge \varphi_1(t), \quad t \in [0,1].$$

PROOF. From (9) and (10), we have

$$\Big(\frac{\varphi_2^{1-\varrho}}{1-\varrho}\Big)'' \le -\frac{\theta}{\varphi_2^{\varrho}} \quad (\varrho \ne 1),$$

$$\Big(\ln(\varphi_2)\Big)'' \le -\frac{\theta}{\varphi_2} \quad (\varrho = 1),$$

and

$$\Big(\frac{\varphi_1^{1-\varrho}}{1-\varrho}\Big)'' \ge -\frac{\theta}{\varphi_1^{\varrho}} \quad (\varrho \ne 1),$$

$$\Big(\ln(\varphi_1)\Big)'' \ge -\frac{\theta}{\varphi_1} \quad (\varrho = 1).$$

Combining the above inequalities, we obtain

$$w'' \le \theta\Big(\frac{1}{\varphi_1^{\varrho}} - \frac{1}{\varphi_2^{\varrho}}\Big), \quad 0 < t < 1, \tag{11}$$

where $w : [0,1] \to \mathbb{R}$ is defined as follows: $w = \frac{\varphi_2^{1-\varrho}}{1-\varrho} - \frac{\varphi_1^{1-\varrho}}{1-\varrho}$ if $\varrho \ne 1$; $w = \ln(\varphi_2) - \ln(\varphi_1)$ if $\varrho = 1$. Clearly, $w \in C^2(0,1) \cap C[0,1]$.

To prove the proposition, we argue by contradiction and assume that there exists a point t_0 of $(0,1)$ such that $\varphi_2(t_0) - \varphi_1(t_0) < 0$. From the assumption, it is easy to see that w reaches a minimum at some point t_* of $(0,1)$ such that

$$w(t_*) = \min_{t\in[0,1]} w(t) < 0, \tag{12}$$

$$w''(t_*) \ge 0. \tag{13}$$

Combining (13) with (11), we have

$$\theta\Big(\frac{1}{\varphi_1^{\varrho}(t_*)} - \frac{1}{\varphi_2^{\varrho}(t_*)}\Big) \ge 0.$$

This implies $\varphi_2(t_*) \ge \varphi_1(t_*)$. However, from (12) we find that $\varphi_2(t_*) < \varphi_1(t_*)$, a contradiction. This completes our proof.

LEMMA 3. Under the assumptions of Theorem 1, for all $\varepsilon \in (0,1)$ there exists a positive constant C independent of ε such that

$$\varphi_\varepsilon(t) \ge C[t(1-t) + \varepsilon^{1/2}]^2, \quad t \in [0,1].$$

PROOF. Let $w_\varepsilon = C[t(1-t) + \varepsilon^{1/2}]^2$, where $C \in (0,1]$ will be determined later. By Proposition 2 and noticing (7), it suffices to show that

$$-w_\varepsilon'' + \lambda\frac{|w_\varepsilon'|^2}{w_\varepsilon} - d_m \min_{[0,1]} f \le 0, \quad t \in (0,1), \tag{14}$$

for some sufficiently small positive constant C independent of ε. Simple calculation shows that

$$w_\varepsilon' = 2C[t(1-t) + \varepsilon^{1/2}](1-2t),$$

$$w_\varepsilon'' = 2C(1-2t)^2 - 4C[t(1-t) + \varepsilon^{1/2}],$$

and

$$\begin{aligned}-w_\varepsilon''+\lambda\frac{|w_\varepsilon'|^2}{w_\varepsilon}-d_m\min_{[0,1]}f=&-2C(1-2t)^2+4C[t(1-t)+\varepsilon^{1/2}]\\&+4C\lambda(1-2t)^2-d_m\min_{[0,1]}f\\ \le&4C\Big(2+\lambda\Big)-d_m\min_{[0,1]}f,\quad 0<t<1.\end{aligned}$$

Choosing a positive constant C such that

$$C\le\min\left\{1,\frac{d_m\min_{[0,1]}f}{4(2+\lambda)}\right\},$$

we find that (14) holds. This completes our proof.

From (7), (8), Lemma 1 and Lemma 3, we derive that for any $\delta\in(0,1/2)$ there exists a positive constant C_δ independent of ε such that

$$|\varphi_\varepsilon''(t)|\le C_\delta,\quad \delta\le t\le 1-\delta. \tag{15}$$

Differentiating (4) with respect to t we get

$$\begin{aligned}\varphi_\varepsilon'''=&\lambda\frac{2\varphi_\varepsilon\varphi_\varepsilon'\varphi_\varepsilon''-(\varphi_\varepsilon')^3}{\varphi_\varepsilon^2}-2mf(t)(1+|\varphi_\varepsilon'|^2)^{m-1}\varphi_\varepsilon'\varphi_\varepsilon''\\&-f'(t)(1+|\varphi_\varepsilon'|^2)^m,\quad 0<t<1.\end{aligned}$$

By (15), Lemma 1 and Lemma 3, we derive that for any $\delta\in(0,1/2)$, there exists a positive constant C_δ independent of ε such that

$$|\varphi_\varepsilon'''(t)|\le C_\delta,\quad \delta\le t\le 1-\delta.$$

From this and Lemma 1 and using Alzelá-Ascoli theorem and diagonal sequential process, we see that there exist a subsequence $\{\varphi_{\varepsilon_n}\}$ of $\{\varphi_\varepsilon\}$ and a function $\varphi\in C^2(0,1)\cap C[0,1]$ such that, as $\varepsilon_n\to 0$,

$$\varphi_{\varepsilon_n}\to\varphi,\quad \text{uniformly in}\quad C[0,1],$$

$$\varphi_{\varepsilon_n}\to\varphi,\quad \text{uniformly in}\quad C^2[\delta,1-\delta].$$

Combining these with (4) and the boundary conditions satisfied by φ_{ε_n}, we find that φ satisfies (1) and (2). By Lemma 3, we have

$$\varphi(t)\ge C[t(1-t)]^2,\quad t\in[0,1], \tag{16}$$

therefore $\varphi>0$ in (0,1), and thus φ is a solution to BVP (1), (2). This completes the proof of Theorem 1.

PROOF OF THEOREM 2. From Theorem 1, we see that for any $\lambda>0$, BVP (1),(2) admits a solution φ which can be approximated by φ_{ε_n} satisfying (4) (or (5)) with $\varepsilon=\varepsilon_n$. Hence it suffices to show φ satisfies $\varphi'(1)=\varphi'(0)=0$ for $\lambda>\frac{1}{2}$. We claim that if $\lambda>\frac{1}{2}$, then there exist positive constants C independent of ε_n such that

$$\varphi_{\varepsilon_n}(t)\le C(1+\varepsilon_n^{1/2}-t)^2\quad\text{on}\quad[0,1], \tag{17}$$

$$\varphi_{\varepsilon_n}(t) \le C(t+\varepsilon_n^{1/2})^2 \quad \text{on} \quad [0,1]. \tag{18}$$

We first show (17). Let $v_{\varepsilon_n} = C(1+\varepsilon_n^{1/2}-t)^2$, where $C \ge 1$ will be determined later. A calculation shows that

$$-v''_{\varepsilon_n} + \lambda \frac{|v'_{\varepsilon_n}|^2}{v_{\varepsilon_n}} - D_m \max_{[0,1]} f = 2C\Big(2\lambda - 1\Big) - D_m \max_{[0,1]} f, \quad 0<t<1.$$

Choosing a positive constant C such that

$$C \ge \max\left\{1, \frac{D_m \max_{[0,1]} f}{2(2\lambda-1)}\right\}$$

and noticing $\lambda > \frac{1}{2}$, we find that

$$-v''_{\varepsilon_n} + \lambda \frac{|v'_{\varepsilon_n}|^2}{v_{\varepsilon_n}} - D_m \max_{[0,1]} f \ge 0, \quad 0<t<1,$$

and then, by Proposition 2 and (8), we obtain (17). Similarly the claim (18) can be proved. Letting $\varepsilon_n \to 0$ in (17) and (18) to yield

$$\varphi(t) \le C \min\{t^2, (1-t)^2\}, \quad t \in [0,1].$$

Combining this with (16) we immediately obtain $\varphi'(1) = \varphi'(0) = 0$. Thus the proof of Theorem 2 is completed.

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References

[1] R. P. Agarwal and D. O'Regan, Singular boundary value problems for superlinear second order ordinary and delay differential euqations, J. Diff. Equ., 130(1996), 333-355.

[2] D. O'Regan, Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1994.

[3] D. O'Regan, Existence Theory for Nonlinear Differential Equations, Kluwer Acad., Dordrecht/Boston/London 1997.

[4] A. Tineo, Existence theorems for a singular two points Drichlet porblem, Nonlinear Anal., 19(1992), 323–333.

[5] J. Wang and J. Jiang, The existence of positive solutions to a singular nonlinear boundary value porblem, J. Math. Anal. Appl., 176(1993), 322–329.

[6] S. Staněk, Positive solutions for singular semipositone boundary value porblems, Math. Comput. Model., 33(2001), 341–351.

[7] D. Q. Jiang, Upper and lower solutions method and a singular boundary value problem, Z. Angew. Math. Mech., 82(7) (2002), 481–490.

[8] D. Bonheure, J. M. Gomes and L. Sanchez, Positive solutions of a second order singular ordinary differential euqation, Nonlinear Anal. TMA., 61(2005), 1383–1399.

[9] M. Bertsch and M. Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation, Nonlinear Anal., 14(1990) 571–592.

[10] W. S. Zhou and X. D. Wei, Positive solutions to BVPs for a singular differential equation, Nonlinear Anal. TMA., 67(2007), 609–617.

[11] Y. Z. Chen and L. C. Wu, Second Order Elliptic Equations and Elliptic Systems, Science Press, Beijing, 1997. (in Chinese) English edition is translated from the 1991 Chinese original by Bei Hu. Translations of Mathematical Monographs, 174. American Mathematical Society, Providence, RI, 1998.