The Holditch Sickles For The Open Homothetic Motions∗

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Abstract

A. Tutar and N. Kuruoğlu [1] had given the following theorem as a generalization of the classical Holditch Theorem [2]: During the closed planar homothetic motions with the period $T$, if the chord $AB$ of fixed length $a + b$ is moved around once on an oval $k_0$, then a point $X \in AB$ ($a = AX, b = BX$) describes a closed path $k_0(X)$ and the “Holditch Ring”, which is bounded by $k_0$ and $k_0(X)$ has the surface area

$$F = h^2(t_0) \pi ab,$$

for $\exists t_0 \in [0, T]$. In this paper, under the open homothetic motions we expressed the Holditch Sickle such that the closed oval is replaced by the boundary of an bounded convex domain and so, the Holditch Sickles given by H. Pottmann [3] for one-parameter Euclidean motions generalized to the homothetic motions.

1 Introduction

Let $E$ and $E'$ be moving and fixed Euclidean planes and $\{O; e_1, e_2\}$ and $\{O'; e_1', e_2'\}$ be their coordinate systems, respectively. By taking $OO' = u = u_1 e_1 + u_2 e_2$, for $u_1, u_2 \in \mathbb{R}$, the motion defined by the transformation

$$x' = hx - u \tag{1}$$

is called one-parameter planar homothetic motion with the homothetic scale $h(t)$ and denoted by $H_1 = E/E'$, where $x$ and $x'$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X \in E$, respectively. Furthermore, at the initial time $t = 0$ the coordinate systems are coincident. Taking $\varphi = \varphi(t)$ as the rotation angle between $e_1$ and $e_1'$, the equation

$$e_1 = \cos \varphi e_1' + \sin \varphi e_2'$$
$$e_2 = -\sin \varphi e_1' + \cos \varphi e_2' \tag{2}$$

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can be written. Also, the homothetic scale \( h \), the rotation angle \( \varphi \) and \( u_1, u_2 \) are continuously differentiable functions of a real time parameter \( t \). If
\[
\begin{align*}
  h(t + T) &= h(t), \quad \varphi(t + T) = \varphi(t) + 2\pi \nu, \quad \forall t \in [0, T], \\
  u_j(t + T) &= u_j(t), \quad j = 1, 2
\end{align*}
\]
then \( H_1 \) is called one-parameter closed planar homothetic motion with the period \( T > 0 \) and the rotation number \( \nu \in \mathbb{Z} \). Otherwise, \( H_1 \) is called one-parameter open planar homothetic motion.

2 Holditch Sickles under Homothetic Motions

During the open homothetic motion \( H_1 \), let unbounded convex curve \( k_o \), be the common orbit curve of the points \( A \) and \( B \) of the moving plane \( E \) and the points \( A \) and \( B \) move to infinity for \( t \to \mp \infty \). Also, there could be a pair of different, parallel tangents \( t_1, t_2 \) of \( k_o \), which is the edge of an unbounded convex domain \( K_o \subset E' \). If there exist contact points \( R_i \) of \( t_i \) with \( k_o \), there exists half lines \( h_i \subset t_i \) of \( k_o \). The distance \( \Delta \) between \( t_1 \) and \( t_2 \) is defined as “wide” of \( K_o \). If there aren’t parallel tangent pairs, then we assume that \( \Delta = +\infty \). Under the open \( H_1 \), let the endpoints of \( AB \) pass through the edge \( k_o \). This is always possible for \( AB' < \Delta \). If \( AB' = \Delta < \infty \), then the desired motion is possible when the contact points \( R_i \) of \( t_i \) are exist. For \( AB' > \Delta \), the motion is impossible. The points \( A \) and \( B \) can turn back in some cases, during the open motion \( H_1 \). The dead centre of an endpoint of \( AB' \) is a instantaneous rotation pole center at the same time. Also, for each position of the chord \( AB \) with fixed lengh \( a + b \), there exists a parallel tangent of \( k_o \) which during the motion makes a complete turn around the total rotation angle \( \delta \). Therefore, the total rotation angle \( \delta \in \mathbb{R}^+ \) of the open \( H_1 \) coincides with tangent rotation angle of \( k_o \).

THEOREM 1. Let \( k_o \) be the edge of unbounded convex domain \( K_o \subset \mathbb{R}^2 \) and \( \delta \in \mathbb{R}^+ \) be its tangent rotation angle. If we assume that the endpoints \( A \) and \( B \) of the straight line \( s \) with length \( a + b \) move from the fixed any point along the curve \( k_0 \) first to the positive direction and then to the negative direction, the point \( X \in s \) \( (a = \overline{AX}, \ b = \overline{BX}) \) describes a curve \( k_o(X) \) and “the Holditch- Sickle” \( S_o \subset K_o \), which is bounded by \( k_o \) and \( k_o(X) \) has the surface area \( F_S = h^2(t_0)\delta(t)/2 \).

PROOF. Let the points \( A = (0, 0), B = (a + b, 0), X = (a, 0) \in E \) have the position \( A', B', X' \) in fixed plane \( E' \) for \( t > 0 \) and analog the position \( A^{-1}, B^{-1}, X^{-1} \) for \( -t \). Then, these two position can coincide with a rotation round a definite centre \( D \in E' \). If the motion \( H_1 = E/E' \) is restricted to time interval \( [-t, t] \), then an open motion \( \tilde{H}_1(t) \) with total rotation angle \( \tilde{\delta}(t) \) is obtained. Under \( \tilde{H}_1(t) \), the region determined by the center \( D \in E' \) and the orbit curve part \( \tilde{k}_o(Y) \) of the fixed point \( Y = (y_1, y_2) \in E \) has the surface area
\[
F^D_Y = F^D_A + \frac{h^2(t_0)\tilde{\delta}(t)}{2} \left( y_1^2 + y_2^2 - \lambda_1 y_1 - \lambda_2 y_2 \right) + \mu_1 y_1 + \mu_2 y_2, \quad [4]
\]

\[\text{Using the mean-value theorem for the integral } \int_{-t}^{t} h(t)\dot{\varphi}(t)dt \text{ which occurs during the calculation of the surface area } F^D_Y \text{ under the motion } \tilde{H}_1(t) \text{ (we assume that the functions } h \text{ and } \varphi \text{ have the same sign in the interval } [-t, t], \text{ we get } t_0 \in [-t, t] \text{ satisfying the equation } h^2(t_0)\int_{-t}^{t} \dot{\varphi}(t)dt = h^2(t_0)\tilde{\delta}(t).\]
for \( \exists t_0 \in [0, T] \) and \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R} \).

The orbit curve parts \( \tilde{k}_o(A) \), \( \tilde{k}_o(X) \) of the points \( A, X \) and the line segments \( A^tX^t, A^{-t}X^{-t} \) describe a closed domain. This closed domain has the oriented surface area

\[
F_{AX}(t) = F_A^D - F_X^D
\]  

(4)

in order \( A^{-t}A^tX^{-t}A^{-t} \). Similarly, we can write

\[
F_{AB}(t) = F_A^D - F_B^D.
\]  

(5)

From eqns. (3), (4) and (5), we get

\[
F_{AX}(t) = \frac{h^2(t_0)ab\tilde{\delta}(t)}{2} + \frac{a}{a+b}F_{AB}(t).
\]  

(6)

Let \( \alpha_A(t) \) and \( \alpha_B(t) \) (resp. \( \alpha_A(-t) \) and \( \alpha_B(-t) \)) be the angles between the chord \( A^tX^t \) (resp. \( A^{-t}X^{-t} \)) and \( k_0 \). Then, for \( F_{AB}(t) \), which is composed of two oriented areas formed by the chord \( A^tX^t \) and the curve \( k_0 \); and the same at \( -t \), we can write for sufficiently large \( t \)

\[
|F_{AB}(t)| \leq (a + b)^2[h^2(t)(\sin\alpha_A(t) + \sin\alpha_B(t)) + h^2(-t)(\sin\alpha_A(-t) + \sin\alpha_B(-t))].
\]

Thus, for \( t \to +\infty \), we have

\[
\lim_{t \to \infty} F_{AB}(t) = 0.
\]  

(7)

Hence, from eqns. (6) and (7), we get

\[
F_S = h^2(t_0)ab\delta/2,
\]  

(8)

where

\[
\lim_{t \to \infty} F_{AX}(t) = F_S, \quad \lim_{t \to \infty} \tilde{\delta}(t) = \delta.
\]

SPECIAL CASE 1. In the case of the homothetic scale \( h \equiv 1 \), we have

\[
F_S = ab\delta/2
\]

which was given by Pottmann [3].

3 Spatial Holditch-Sickles

Let \( k_A \) and \( k_B \) be normal cross-sections of \( C^1 \)-cylinder \( \Gamma \subset \mathbb{R}^3 \) (which is the edge of an unbounded convex domain in \( \mathbb{R}^3 \)) with the planes \( z = 0 \) and \( z = k \). Let the endpoints \( A, B \) of a straight line \( s \) with constant length \( l \) move along congruent convex curves \( k_A \) and \( k_B \) with tangent rotation angle \( \delta \). Then, the straight line \( s \) describes a ruled surface whose rulings have the constant angle \( \beta = \arcsin(k/hl) \) with the planes \( z = 0 \) and \( z = k \).
The region we defined as spatial Holditch Sickle $S \subset \mathbb{R}^3$ is the point set bounded by the ruled surface and the cylinder parts between the planes $z = 0$ and $z = k$. During the motion of $s$, the points $X \in s$ draw planar curves on planes $z = c$, $(0 \leq c \leq k)$. The cross-sections of spatial Holditch-Sickles $S$ in planes $z = c$ is planar Holditch-Sickles with the surface area

$$F_S(c) = h^2(t_0) \frac{\delta}{2} c(k - c) \cot^2 \beta.$$  \hspace{1cm} (9)

Then, using Cavalieri-Principle, the volume $V_S$ of $S$ is

$$V_S = \int_0^k F_S(z) dz.$$  \hspace{1cm} (10)

So, using eqns. (9) and (10), we can give the following theorem:

**THEOREM 2.** The volume of spatial Holditch-Sickles $S$ with the height $k$, the total rotation angle $\delta$ and the slope angle $\beta$ is

$$V_S = h^2(t_0) \frac{\delta}{2} k^3 \cot^2 \beta.$$  

**SPECIAL CASE 2.** In the case of the homothetic scale $h \equiv 1$, we have

$$V_S = \frac{\delta}{2} k^3 \cot^2 \beta,$$

which was given by Pottmann [3].

**References**


