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Unilateral Monotone Iteration Scheme For A Forced Duffing Equation With Periodic Boundary Conditions*

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Abstract

In this paper, we apply the generalized quasilinearization technique to a forced Duffing equation with periodic boundary conditions and obtain a monotone sequence of approximate solutions converging quadratically to the unique solution of the problem.

1 Introduction

An interesting and fruitful technique for proving existence results for nonlinear problems is the method of upper and lower solutions. This method coupled with the monotone iterative technique, known as quasilinearization technique [1], manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. However, the concavity/convexity assumption that is demanded by the method of quasilinearization, proved to be a stumbling block for further development of the theory. The nineties brought new dimensions to this technique. The most interesting new idea was introduced by Lakshmikantham [2-3] who generalized the method of quasilinearizaion by relaxing the convexity assumption. This development was so significant that it attracted the attention of many researchers and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, see [4-14].

In this paper, we consider a nonlinear periodic boundary value problem involving a forced Duffing equation and develop the generalized quasilinearization method for the problem at hand. In fact, we obtain a sequence of lower solutions (approximate solutions) converging monotonically and quadratically to the unique solution of the problem. Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of Duffing equation is in the field of the prediction of diseases. In Section 2, we discuss some basic results while Section 3 is devoted to the main result.

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2 Preliminary Notes

We know that the homogeneous periodic boundary value problem

$$-u''(x) - ku'(x) - \lambda u(x) = 0, \quad x \in [0, \pi],$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi),$$

has only the trivial solution if and only if $\lambda \neq 4n^2$ for $k = 0$ and $\lambda \neq 0$ for $k \neq 0$ for all $n \in \{0, 1, 2, ...\}$. Consequently, for these values of λ and for any $\sigma(x) \in C([0, \pi])$, the non homogeneous problem

$$-u''(x) - ku'(x) - \lambda u(x) = \sigma(x), \quad x \in [0, \pi]$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi),$$

has a unique solution

$$u(x) = \int_0^\pi G_\lambda(x, y)\sigma(y)dy,$$

where $G_\lambda(x, y)$ is the Green's function given by

$$G_\lambda(x,y) = \begin{cases} \xi_1 \left(\sinh\left(\frac{\sqrt{k^2-4\lambda}}{2}\right)(\pi - (y - x)) + e^{\frac{-k}{2}\pi} \sinh\left(\frac{\sqrt{k^2-4\lambda}}{2}\right)(y - x)\right), & 0 \leq x \leq y, \\ \xi_1 \left(\sinh\left(\frac{\sqrt{k^2-4\lambda}}{2}\right)(\pi - (x - y)) + e^{\frac{k}{2}\pi} \sinh\left(\frac{\sqrt{k^2-4\lambda}}{2}\right)(x - y)\right), & y \leq x \leq \pi, \end{cases}$$

for $\lambda < \frac{k^2}{4}$ and

$$G_\lambda(x,y) = \begin{cases} \xi_2 \left(\sin\left(\frac{\sqrt{4\lambda-k^2}}{2}\right)(\pi - (y - x)) + e^{\frac{-k}{2}\pi} \sin\left(\frac{\sqrt{4\lambda-k^2}}{2}\right)(y - x)\right), & 0 \leq x \leq y, \\ \xi_2 \left(\sin\left(\frac{\sqrt{4\lambda-k^2}}{2}\right)(\pi - (x - y)) + e^{\frac{k}{2}\pi} \sin\left(\frac{\sqrt{4\lambda-k^2}}{2}\right)(x - y)\right), & y \leq x \leq \pi, \end{cases}$$

for $\lambda > \frac{k^2}{4}$, where

$$\xi_1 = \frac{2e^{\frac{-k}{2}(x+\pi-y)}}{\sqrt{k^2 - 4\lambda}(2e^{\frac{-k}{2}\pi}\cosh(\frac{\sqrt{k^2-4\lambda}}{2}\pi) - 1 - e^{-k\pi})},$$

$$\xi_2 = \frac{-2e^{\frac{-k}{2}(x+\pi-y)}}{\sqrt{4\lambda - k^2}(1 + e^{-k\pi} - 2e^{\frac{-k}{2}\pi}\cos(\frac{\sqrt{4\lambda-k^2}}{2})\pi)}.$$

We note that $G_\lambda(x, y) > 0$ for $\lambda < 0$ and $G_\lambda(x, y) \leq 0$ for $\frac{k^2}{4} < \lambda \leq \frac{k^2}{4} + 1$.

Now, we consider the following nonlinear periodic boundary value problem (PBVP)

$$-u''(x) - ku'(x) = f(x, u(x)), \quad x \in [0, \pi], \tag{1}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi), \tag{2}$$

where $f \in [0, \pi] \times R \to R$ is continuous.

We say that $\alpha \in C^2([0,\pi])$ is a lower solution of (1)-(2) if

$$-\alpha''(x) - k\alpha'(x) \le f(x,\alpha(x)), \quad x \in [0,\pi],$$

$$\alpha(0) = \alpha(\pi), \quad \alpha'(0) \ge \alpha'(\pi).$$

Similarly, $\beta \in C^2([0,\pi])$ is an upper solution of (1)-(2) if

$$-\beta''(x) - k\beta'(x) \ge f(x,\beta(x)), \quad x \in [0,\pi],$$

$$\beta(0) = \beta(\pi), \quad \beta'(0) \le \beta'(\pi).$$

In order to prove the main result, we need the following theorem. We omit the proof of this theorem as it is based on the standard arguments [5].

THEOREM 1. Suppose that $\alpha, \beta \in C^2([0,\pi], R)$ are lower and upper solutions of (1)-(2) respectively such that $\alpha(x) \le \beta(x), \quad \forall x \in [0,\pi]$. Then there exists at least one solution $u(x)$ of (1)-(2) such that $\alpha(x) \le u(x) \le \beta(x)$ on $[0,\pi]$.

3 Generalized Quasilinearization Method

We have the following result.

THEOREM 2. Assume that

($\mathbf{A_1}$) $\alpha, \beta \in C^2([0,\pi], R)$ are lower and upper solutions of (1)-(2) such that $\alpha(x) \le \beta(x)$ on $[0,\pi]$.

($\mathbf{A_2}$) $f \in C^2([0,\pi] \times R)$ and $\frac{\partial f}{\partial u}(x,u) < 0$ for every $(x,u) \in S$, where

$$S = \{(x,u) \in R^2 : x \in [0,\pi] \text{ and } u \in [\alpha(x), \beta(x)]\}.$$

Then there exists a monotone sequence $\{q_n\}$ which converges uniformly and quadratically to a unique solution of (1)-(2).

PROOF. In view of the assumption (A_2) and the mean value theorem, we have

$$f(x,u) \ge f(x,v) + [\frac{\partial}{\partial u} f(x,v) + 2mv](u-v) - m(u^2 - v^2), \quad m > 0,$$

for every $x \in [0,\pi]$ and $u, v \in R$ such that $\alpha(x) \le v \le u \le \beta(x)$ on $[0,\pi]$. Here, we have used $\frac{\partial^2 f}{\partial u^2}(x,u) \ge -2m, \quad (x,u) \in S$, which follows from (A_2). We define the function $g(x,u,v)$ as

$$g(x,u,v) = f(x,v) + [\frac{\partial}{\partial u} f(x,v) + 2mv](u-v) - m(u^2 - v^2).$$

Observe that

$$g(x,u,v) \le f(x,u), \quad g(x,u,u) = f(x,u). \tag{3}$$

It follows from (A_2) and (3) that $g(x,u,v)$ is strictly decreasing in u for each fixed $(x,v) \in [0,\pi] \times R$ and satisfies one sided Lipschitz condition

$$g(x,u_1,v) - g(x,u_2,v) \leq L(u_1 - u_2), \quad L > 0. \tag{4}$$

Now, we set $\alpha = q_0$ and consider the PBVP

$$-u(x) - ku'(x) = g(x,u(x),q_0(x)), \quad x \in [0,\pi] \tag{5}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi). \tag{6}$$

In view of (A_1) and (3), we have

$$-q_0''(x) - kq_0'(x) \leq f(x,q_0(x)) = g(x,q_0(x),q_0(x)), \quad x \in [0,\pi],$$

$$q_0(0) = q_0(\pi), \quad q_0'(0) \geq q_0'(\pi),$$

and

$$-\beta''(x) - k\beta'(x) \geq f(x,\beta(x)) \geq g(x,\beta(x),q_0(x)), \quad x \in [0,\pi],$$
$$\beta(0) = \beta(\pi), \quad \beta'(0) \leq \beta'(\pi),$$

which imply that $q_0(x)$ and $\beta(x)$ are lower and upper solutions of (5)-(6) respectively. Hence, by Theorem 2 and (4), there exists a unique solution $q_1(x)$ of (5)-(6) such that

$$q_0(x) \leq q_1(x) \leq \beta(x), \quad \text{on} \quad [0,\pi].$$

Next, consider the PBVP

$$-u(x) - ku'(x) = g(x,u(x),q_1(x)), \quad x \in [0,\pi], \tag{7}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi). \tag{8}$$

using (A_1) and employing the fact that $q_1(x)$ is a solution of (5)-(6), we obtain

$$-q_1''(x) - kq_1'(x) = g(x,q_1(x),q_0(x)) \leq g(x,q_1(x),q_1(x)), \quad x \in [0,\pi], \tag{9}$$

$$q_1(0) = q_1(\pi), \quad q_1'(0) \geq q_1'(\pi), \tag{10}$$

and

$$-\beta''(x) - k\beta'(x) \geq f(x,\beta(x)) \geq g(x,\beta(x),q_1(x)), \quad x \in [0,\pi], \tag{11}$$
$$\beta(0) = \beta(\pi), \quad \beta'(0) \leq \beta'(\pi), \tag{12}$$

From (9)-(10) and (11)-(12), we find that $q_1(x)$ and $\beta(x)$ are lower and upper solutions of (7)-(8) respectively. Again,by Theorem 2 and (4), there exists a unique solution $q_2(x)$ of (7)-(8) such that

$$q_1(x) \leq q_2(x) \leq \beta(x) \quad \text{on} \quad [0,\pi].$$

This process can be continued successively to obtain a monotone sequence $\{q_n(x)\}$ satisfying

$$q_0(x) \leq q_1(x) \leq .... \leq q_{n-1}(x) \leq q_n(x) \leq \beta(x), \quad \text{on} \quad [0,\pi],$$

where the element $q_n(x)$ of the sequence $\{q_n(x)\}$ is a solution of the problem

$$-u(x) - ku'(x) = g(x, u(x), q_{n-1}(x)), \quad x \in [0, \pi]$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi).$$

Since the sequence $\{q_n\}$ is monotone, it follows that it has a pointwise limit $q(x)$. To show that $q(x)$ is in fact a solution of (1)-(2), we note that $q_n(x)$ is a solution of the problem

$$-u''(x) - ku'(x) - \lambda u(x) = \Psi_n(x), \quad x \in [0, \pi], \tag{13}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi), \tag{14}$$

where $\Psi_n(x) = g(x, q_n(x), q_{n-1}(x)) - \lambda q_n(x)$ for every $x \in [0, \pi]$. Since $g(x, u, v)$ is continuous on S and $\alpha(x) \le q_n(x) \le \beta(x)$ on $[0, \pi]$, it follows that $\{\Psi_n(x\})$ is bounded in $C[0, \pi]$. Thus, $q_n(x)$, the solution of (13)-(14) can be written as

$$q_n(x) = \int_0^\pi G_\lambda(x, y)\Psi_n(x)dy. \tag{15}$$

This implies that $\{q_n(x)\}$ is bounded in $C^2([0, \pi])$ and hence $\{q_n(x)\} \nearrow q(x)$ uniformly on $[0, \pi]$. Consequently, taking limit $n \to \infty$ of (15) yields

$$q(x) = \int_0^\pi G_\lambda(x, y)[f(y, q(y)) - \lambda q(y)]dy, \quad x \in [0, \pi].$$

Thus, we have shown that $q(x)$ is a solution of (1)-(2).

Now, we prove that the convergence of the sequence is quadratic. For that, we define

$$F(x, u) = f(x, u) + mu^2.$$

In view of (A_2), we can find a constant C such that

$$0 \le \frac{\partial^2}{\partial u^2} F(x, u) \le C.$$

Letting $e_n(x) = q(x) - q_n(x), \quad n = 1, 2, 3, ...,$ we have

$$\begin{aligned}
-e_n''(x) - ke_n'(x) &= q_n''(x) + kq_n'(x) - q''(x) - kq'(x) \\
&= f(x, q(x)) - f(x, q_{n-1}(x)) \\
&\quad - \left[\frac{\partial}{\partial u} f(x, q_{n-1}(x)) + 2mq_{n-1}(x)\right](q_n(x) - q_{n-1}(x)) \\
&\quad + m(q_n^2(x) - q_{n-1}^2(x)) \\
&= F(x, q(x)) - mq^2(x) - F(x, q_{n-1}(x)) \\
&\quad + mq_{n-1}^2(x) - \frac{\partial}{\partial u} F(x, q_{n-1}(x))(q_n(x) - q_{n-1}(x)) \\
&\quad + m(q_n^2(x) - q_{n-1}^2(x)) \\
&= F(x, q(x)) - F(x, q_{n-1}(x)) \\
&\quad - \frac{\partial}{\partial u} F(x, q_{n-1}(x))(q_n(x) - q_{n-1}(x)) - m(q^2(x) - q_n^2(x)),
\end{aligned}$$

$$e_n(0) = e_n(\pi), \quad e_n'(0) = e_n'(\pi).$$

Using the mean value theorem repeatedly, we obtain

$$\begin{aligned}
-e_n''(x) - ke_n'(x) &= \frac{\partial}{\partial u}F(x,\xi)(q(x) - q_{n-1}(x)) - \frac{\partial}{\partial u}F(x, q_{n-1}(x))(q_n(x) - q_{n-1}(x)) \\
&\quad -m(q^2(x) - q_n^2(x)) \\
&= \frac{\partial}{\partial u}F(x,\xi)(q(x) - q_{n-1}(x)) - \frac{\partial}{\partial u}F(x, q_{n-1}(x))(q(x) - q_{n-1}(x)) \\
&\quad -\frac{\partial}{\partial u}F(x, q_{n-1}(x))(q_n(x) - q(x)) - m(q^2(x) - q_n^2(x)) \\
&= \left[\frac{\partial}{\partial u}F(x,\xi(x)) - \frac{\partial}{\partial u}F(x, q_{n-1}(x))\right](q(x) - q_{n-1}(x)) \\
&\quad + \left[\frac{\partial}{\partial u}F(x, q_{n-1}(x)) - m(q(x) + q_n(x))\right](q - q_n(x)) \\
&= \frac{\partial^2}{\partial u^2}F(x,\eta)(\xi(x) - q_{n-1}(x))e_{n-1}(x) \\
&\quad + \left[\frac{\partial}{\partial u}F(x, q_{n-1}(x)) - m(q(x) + q_n(x))\right]e_n(x),
\end{aligned}$$

$$e_n(0) = e_n(\pi), \quad e_n'(0) = e_n'(\pi).$$

where $q_{n-1}(x) \leq \eta(x) \leq \xi(x) \leq q(x)$ on $[0, \pi]$ (η and ξ also depend on $q_{n-1}(x)$ and $q(x)$). Substituting

$$a_n(x) = \frac{\partial}{\partial u}F(x, q_{n-1}(x)) - m(q(x) + q_n(x)),$$

$$b_n(x) + Ce_{n-1}^2(x) = \frac{\partial^2}{\partial u^2}F(x, \eta(x))e_{n-1}(x)(\xi - q_{n-1}(x)),$$

in the above expression gives $b_n(x) \leq 0$ on $[0, \pi]$ and

$$-e_n''(x) - ke_n'(x) - e_n(x)a_n(x) = Ce_{n-1}^2(x) + b_n(x), \;\; x \in [0, \pi],$$

$$e_n(0) = e_n(\pi), \quad e_n'(0) = e_n'(\pi).$$

Since $\lim_{n\to\infty} a_n(x) = \frac{\partial F}{\partial u}(x, q(x)) - 2mq(x) = \frac{\partial f}{\partial u}(x, q(x))$ and $\frac{\partial f}{\partial u}(x, q(x)) < 0$, therefore, for $\lambda < 0$, there exists $n_0 \in N$ such that for $n \geq n_0$, we have $a_n(x) < \lambda < 0, x \in [0, \pi]$. Therefore,the error function $e_n(x)$ satisfies the following problem

$$-e_n''(x) - ke_n'(x) - \lambda e_n(x) = (a_n(x) - \lambda)e_n(x) + Ce_{n-1}^2(x) + b_n(x), x \in [0, \pi],$$

whose solution is

$$e_n(x) = \int_0^\pi G_\lambda(x, y)[(a_n(y) - \lambda)e_n(y) + Ce_{n-1}^2(y) + b_n(y)]dy.$$

Since $a_n(y) - \lambda < 0, b_n(y) \leq 0$, and $G_\lambda(x, y) > 0$ for $\lambda < 0$, therefore, it follows that

$$G_\lambda(x, y)[(a_n(y) - \lambda)e_n(y) + Ce_{n-1}^2(y) + b_n(y)] < G_\lambda(x, y)Ce_{n-1}^2(y).$$

Thus, we obtain

$$0 \leq e_n(x) \leq C \int_0^{\pi} G_\lambda(x,y) e_{n-1}^2(y) dy,$$

which can be expressed as

$$\|e_n\| \leq C_1 \|e_{n-1}\|^2,$$

where $C_1 = C \max \int_0^{\pi} G_\lambda(x,y)dy$ and $\|e_n\| = \max\{|e_n| : x \in [0,\pi]\}$ is the usual uniform norm.

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