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Singular Integrals Of The Time Harmonic Maxwell Equations Theory On A Piecewise Liapunov Surface*

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Abstract

We give a short proof of a formula of Poincaré-Bertrand in the setting of theory of time-harmonic electromagnetic fields on a piece-wise Liapunov surface, as well as for some versions of quaternionic analysis.

1 Introduction

Let Γ be a closed Liapunov curve in the complex plane and let f be a Hölder function on $\Gamma \times \Gamma$. Then, everywhere on Γ,

$$\frac{1}{\pi i}\int_{\Gamma_\tau}\frac{d\tau}{\tau - t}\cdot\frac{1}{\pi i}\int_{\Gamma_{\tau_1}}\frac{f(\tau,\tau_1)d\tau_1}{\tau_1-\tau} = f(t,t) + \frac{1}{\pi i}\int_{\Gamma_{\tau_1}} d\tau_1 \cdot \frac{1}{\pi i}\int_{\Gamma_\tau}\frac{f(\tau,\tau_1)d\tau}{(\tau - t)(\tau_1-\tau)}, \quad (1)$$

which is usually called the Poincaré-Bertrand formula, the integrals being understood in the sense of the Cauchy principal value. The Poincaré-Bertrand formula plays a significant role in the theory of one-dimensional singular integral equations with the Cauchy kernel and its numerous applications. Indeed, all the integrals in (1) contain the (singular) Cauchy kernel, and its importance for one-dimensional complex analysis is obvious.

It is known that the theory of solutions of the Maxwell equations reduces, in some degenerate cases, to that of complex holomorphic functions. Hence, one may consider the former to be a generalization of the latter. At the same time, not too many facts from the holomorphic function theory have their extensions onto the Maxwell equations theory. In the present paper we study a number of generalization of (1). In realizing this study we follow the approach first presented in [2] and developed in [3], [8], [11] which are based on the intimate relation between time-harmonic electromagnetic fields and quaternion-valued α-hyperholomorphic functions, see book [3]. This approach proved to be quite efficient and heuristic since it allows to exploit a profound similarity between holomorphic functions in one variable and α-hyperholomorphic functions. The paper is organized as follows. In Section 2 the reader can find the Poincaré-Bertrand formula for time-harmonic electromagnetic fields theory, i.e., theory of solutions of the time-harmonic Maxwell equations. The proof can be found in the Section 6, and is based on

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the contents of Sections 3-5. In Section 4 we present the Poincaré-Bertrand formula for α-hyperholomorphic quaternionic functional theory on a piece-wise Liapunov surface.

Note that the Poincaré-Bertrand formula on closed piece-wise smooth manifold in $\mathbb{C}^n$ for Bochner-Martinelli type singular integrals was studied, for example, by Zhong and Chen [12] as well as Lin and Qiu [6].

2 Time Harmonic Electromagnetic Fields Theory and the Cauchy-Maxwell Integral

Let Ω be a domain, i.e., a connected open set in the three-dimensional Euclidean space $\mathbb{R}^3$, $\Gamma := \partial\Omega$ be its boundary. We consider the following system of *time-harmonic Maxwell equations*:

$$rot\vec{H} = \sigma\vec{E}, \quad rot\vec{E} = i\omega\mu\vec{H}, \tag{2}$$

$$div\vec{H} = 0, \quad div\vec{E} = 0, \tag{3}$$

where $\vec{E}$, $\vec{H} : \Omega \subset \mathbb{R}^3 \to \mathbb{C}^3$; $\sigma := \sigma^* - i\omega\varepsilon$ is a complex electrical conductivity; ε is a dielectric constant; μ is a magnetic permeability; σ^* is a medium electrical conductivity being inverse to its electrical resistivity: $\sigma^* = \frac{1}{\rho}$. If $\vec{E}$ and $\vec{H}$ form a solution of the time-harmonic Maxwell equations in Ω, then $(\vec{E}, \vec{H})$ is called a *time-harmonic electromagnetic field*.

Set

$$\mathcal{M} := \begin{pmatrix} \sigma & -rot \\ rot & -i\omega\mu \end{pmatrix},$$

then the equations (2) become

$$\mathcal{M}\left[\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}\right] = 0.$$

The operator $\mathcal{M}$ acts on the space $C^1(\Omega, \mathbb{C}^3 \times \mathbb{C}^3)$. Recalling equations (3) we will consider, for $k \in \mathbb{N} \cup \{0\}$,

$$\hat{C}^k := \hat{C}^k(\Omega, \mathbb{C}^3 \times \mathbb{C}^3) := \left\{\begin{pmatrix} \vec{f} \\ \vec{g} \end{pmatrix} \in C^k(\Omega, \mathbb{C}^3 \times \mathbb{C}^3) \mid div\ \vec{f} = div\ \vec{g} = 0\right\}.$$

The operator

$$\widehat{\mathcal{M}} := \mathcal{M}_{|\hat{C}^1},$$

i.e., the restriction of $\mathcal{M}$ onto $\hat{C}^1$, will be termed "the time-harmonic Maxwell operator". For more details see, e.g., [3], [4].

The integral

$$K_{\mathcal{M}}[\vec{g}](x) := \int_\Gamma \mathcal{K}_{\mathcal{M}}(\tau, x) \star \vec{g}(\tau) ds, \ x \notin \Gamma,$$

plays the role of the Cauchy-type integral in the theory of time-harmonic electromagnetic fields with $\vec{g} := \begin{pmatrix} \vec{e} \\ \vec{h} \end{pmatrix} : \Gamma \to \mathbb{C}^3 \times \mathbb{C}^3$ (see [8]) being a pair of integrable vector fields and we shall call it the Cauchy-Maxwell-type integral, where $\mathcal{K}_{\mathcal{M}}$ is time-harmonic Cauchy-Maxwell kernel in a formula (11) in [8], definition of "$\star$" can be seen, e.g., in reference [8], ds is an element of the surface area in $\mathbb{R}^3$.

Let $H_\mu(\Gamma, \mathbb{C}^3) := \{\vec{f} \in \mathbb{C}^3 : \ |\vec{f}(t_1) - \vec{f}(t_2)| \leq L_f \cdot |t_1 - t_2|^\mu; \ \forall \{t_1, t_2\} \subset \Gamma, \ L_f = const\}$ denote the class of functions satisfying the Hölder condition with the exponent $0 < \mu \leq 1$. Here $|\vec{f}|$ means the Euclidean norm in $\mathbb{C}^3 = \mathbb{R}^6$ while $|t|$ is the Euclidean norm in $\mathbb{R}^3$. Let Γ be a surface in $\mathbb{R}^3$ which contains a finite number of conical points and a finite number of non-intersecting edges such that none of the edges contain any of conical points. If the complement (in Γ) of the union of conical points and edges, is a Liapunov surface, then we shall refer to Γ as a piece-wise Liapunov surface in $\mathbb{R}^3$.

THEOREM 2.1 (Poincaré-Bertrand formula for time-harmonic electromagnetic theory). Let Ω be a bounded domain in $\mathbb{R}^3$ with the piece-wise Liapunov boundary. Let $\vec{e}, \vec{h} \in H_\mu(\Gamma \times \Gamma; \mathbb{C}^3), 0 < \mu < 1$. Then the following equalities hold, everywhere on Γ:

$$\int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \mathcal{K}_{\mathcal{M}}(t,\tau) \star \left(\mathcal{K}_{\mathcal{M}}(\tau,\tau_1) \star \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau) ds(\tau_1)$$

$$- \int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \mathcal{U}_\alpha(t-\tau) * \left(\mathcal{U}_\alpha(\tau-\tau_1) \diamond \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau) ds(\tau_1) + \gamma^2(t) \begin{pmatrix} \vec{e}(t,t) \\ \vec{h}(t,t) \end{pmatrix}$$

$$= \int_{\Gamma_\tau} \int_{\Gamma_{\tau_1}} \mathcal{K}_{\mathcal{M}}(t,\tau) \star \left(\mathcal{K}_{\mathcal{M}}(\tau,\tau_1) \star \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau_1) ds(\tau)$$

$$- \int_{\Gamma_\tau} \int_{\Gamma_{\tau_1}} \mathcal{U}_\alpha(t-\tau) * \left(\mathcal{U}_\alpha(\tau-\tau_1) \diamond \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau_1) ds(\tau);$$

$$\int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \mathcal{U}_\alpha(t-\tau) \diamond \left(\mathcal{K}_{\mathcal{M}}(\tau,\tau_1) \star \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau) ds(\tau_1)$$

$$+ \int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \langle grad\ \theta_\alpha(t-\tau), \vec{n}_\tau \rangle \left(\mathcal{U}_\alpha(\tau-\tau_1) \diamond \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau) ds(\tau_1)$$

$$= \int_{\Gamma_\tau} \int_{\Gamma_{\tau_1}} \mathcal{U}_\alpha(t-\tau) \diamond \left(\mathcal{K}_{\mathcal{M}}(\tau,\tau_1) \star \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau_1) ds(\tau)$$

$$+ \int_{\Gamma_\tau} \int_{\Gamma_{\tau_1}} \langle grad\ \theta_\alpha(t-\tau), \vec{n}_\tau \rangle \left(\mathcal{U}_\alpha(\tau-\tau_1) \diamond \begin{pmatrix} \vec{e}(\tau_1,\tau) \\ \vec{h}(\tau_1,\tau) \end{pmatrix} \right) ds(\tau_1) ds(\tau),$$

where the integrals being understood in the sense of the Cauchy principal value, $\gamma(t) := \frac{\eta(t)}{4\pi}$; $\eta(t)$ is the measure of a solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t; "$\diamond$" and $\mathcal{U}_\alpha$ were defined in [8].

The proof will be presented in Section 6. Note that if Γ is a Liapunov surface, then this Theorem coincides with the result in paper [8].

3 Basic Facts of Hyperholomorphic Function Theory

In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [1], [3].

Let $\mathbb{H}(\mathbb{C})$ be the set of complex quaternions, it means that each quaternion a is represented in the form $a = \sum_{k=0}^{3} a_k i_k$, with the standard basis $\{i_0 := 1, i_1, i_2, i_3\}$, where $\{a_k : k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup \{0\};\ \mathbb{N}_3 := \{1,2,3\}\} \subset \mathbb{C}$. We use the Euclidean norm $|a|$ in $\mathbb{H}(\mathbb{C})$, defined by $|a| := \sqrt{\sum_{k=0}^{3} |a_k|^2}$.

Let $\lambda \in \mathbb{C}\setminus\{0\}$, and let α be its complex square root: $\alpha \in \mathbb{C},\ \alpha^2 = \lambda$. The function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ is called *left-α-hyperholomorphic* if

$$D_\alpha f := \alpha f + i_1 \frac{\partial}{\partial x_1} f + i_2 \frac{\partial}{\partial x_2} f + i_3 \frac{\partial}{\partial x_3} f = 0.$$

Setting

$$\overline{D}_\alpha f := \alpha f - i_1 \frac{\partial}{\partial x_1} f - i_2 \frac{\partial}{\partial x_2} f - i_3 \frac{\partial}{\partial x_3} f.$$

Let $\alpha \in \mathbb{C}$ and let θ_α be the fundamental solution of the Helmholtz operator $\Delta_\lambda := \Delta + I\lambda$, where $\Delta := \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$ and I is the identity operator. Then the fundamental solution of the operator D_α, $\mathcal{K}_\alpha$, is given by the formula (see [3]):

$$\mathcal{K}_\alpha(x) := -D_\alpha \theta_\alpha(x),$$

and its explicit form can be seen, e.g., in [11]. We shall use the notation $C^p(\Omega, \mathbb{H}(\mathbb{C}))$ for $p \in \mathbb{N} \cup \{0\}$, which has the usual component-wise meaning.

Let $\sigma_x := \sum_{k=1}^{3} (-1)^{k-1} i_k dx_{[k]}$, where $dx_{[k]}$ denotes as usual the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. Let $\Omega = \Omega^+$ be a domain in $\mathbb{R}^3$ with the boundary Γ which is assumed to be a piece-wise Liapunov surface; denote $\Omega^- := \mathbb{R}^3 \setminus (\Omega^+ \cup \Gamma)$. If f is a Hölder function then its α-hyperholomorphic left Cauchy-type integral is defined (see [3]):

$$K_\alpha[f](x) := \int_\Gamma \mathcal{K}_\alpha(\tau - x) \cdot \sigma_\tau \cdot f(\tau),\ x \in \Omega^\pm.$$

For more information about α-hyperholomorphic functions, we refer the reader to [1], [3], [9].

4 The Poincaré-Bertrand Formula for α - Hyperholomorphic Function Theory on a Piecewise Liapunov Surface

We begin with the following result.

LEMMA 4.1. Let Ω be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov surface. For $t \in \Gamma$,

$$\int_\Gamma \mathcal{K}_\alpha(\tau - t)\sigma_\tau = \gamma(t), \tag{4}$$

where the integral being understood in the sense of the Cauchy principal value, $\gamma(t) := \frac{\eta(t)}{4\pi}$; $\eta(t)$ is the measure of a solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t.

PROOF. The proof is a direct computation from the Sokhotski-Plemelj theorem [10, Theorem 2.1] and the Cauchy's integral formula [1, Theorem 3.28], and will be omitted.

LEMMA 4.2. Let Ω be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov surface. Suppose $f(\tau_1,\tau) := f_0(\tau_1,\tau)|\tau_1-\tau|^{-\nu}, 0 \leq \nu < 2$, and $f_0 \in H_\mu(\Gamma\times\Gamma,\mathbb{H}(\mathbb{C}))$. Then the following formula holds for interchange of the order of integration for all $t\in\Gamma$:

$$\int_{\Gamma_\tau}\int_{\Gamma_{\tau_1}} \mathcal{K}_\alpha(\tau-t)\sigma_\tau f(\tau_1,\tau)\sigma_{\tau_1} = \int_{\Gamma_{\tau_1}}\int_{\Gamma_\tau} \mathcal{K}_\alpha(\tau-t)\sigma_\tau f(\tau_1,\tau)\sigma_{\tau_1}.$$

PROOF. The proof is completely analogous to [5, §22], the only need to do is using the (4) instead of $\frac{1}{2}$ for the case of smooth boundary.

LEMMA 4.3. Let Ω be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov surface. If $t,\tau_1\in\Gamma, t\neq\tau_1$ then

$$\int_{\Gamma_\tau} \mathcal{K}_\alpha(\tau-t)\sigma_\tau\mathcal{K}_\alpha(\tau-\tau_1) = 0.$$

PROOF. The argument can be proved by analogy with [7, Lemma 3], taking into account the Sokhotski-Plemelj formulas [11, Theorem 3.1] and the Cauchy's integral formula [1, Theorem 3.28].

THEOREM 4.4 (Poincaré-Bertrand formula for α-hyperholomorphic function theory with $\alpha\in\mathbb{C}$). Let Ω be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov boundary. Assume that $f\in H_\mu(\Gamma\times\Gamma;\mathbb{H}(\mathbb{C}))$, where $0<\mu\leq 1$. Then for all $t\in\Gamma$,

$$\int_{\Gamma_\tau}\int_{\Gamma_{\tau_1}} \mathcal{K}_\alpha(\tau-t)\sigma_\tau\mathcal{K}_\alpha(\tau_1-\tau)\sigma_{\tau_1} f(\tau_1,\tau)$$

$$= \int_{\Gamma_{\tau_1}}\int_{\Gamma_\tau} \mathcal{K}_\alpha(\tau-t)\sigma_\tau\mathcal{K}_\alpha(\tau_1-\tau)\sigma_{\tau_1} f(\tau_1,\tau) + \gamma^2(t)f(t,t). \tag{5}$$

PROOF. The proof is based on Lemmas 4.1, 4.2 and 4.3. It is almost analogous to [8, Theorem 2.7].

5 Function Theory for the Quaternionic Maxwell Operator

We start this Section with a brief description of the relations between the time-harmonic electromagnetic fields theory and the theory of α-hyperholomorphic functions. One can find more about this in [3], [8]. Let $Mat_{2\times 1}$ be its subset consisting of matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ which are identified naturally with columns $\begin{pmatrix} a \\ b \end{pmatrix}$. Abusing a little we shall not distinguish between $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$.

We will consider the following matrix operator

$$\mathcal{N} := \begin{pmatrix} \sigma & -D \\ D & -i\omega\mu \end{pmatrix}$$

on the set $C^1(\Omega, Mat_{2\times 2}(\mathbb{H}(\mathbb{C})))$, $Mat_{2\times 2}(\mathbb{H}(\mathbb{C}))$ being the set of 2×2 matrices with entries from $\mathbb{H}(\mathbb{C})$. Hence for us

$$\mathcal{N} : C^1(\Omega,\ Mat_{2\times 2}(\mathbb{H}(\mathbb{C}))) \to C^0(\Omega,\ Mat_{2\times 2}(\mathbb{H}(\mathbb{C}))).$$

Its restriction $\widetilde{\mathcal{N}} := \mathcal{N}_{|C^1(\Omega,\ Mat_{2\times 1}(\mathbb{C}^3))}$ onto, in fact, pairs of $\mathbb{C}^3$-valued functions has the form

$$\widetilde{\mathcal{N}} = \begin{pmatrix} \sigma & div - rot \\ -div + rot & -i\omega\mu \end{pmatrix}$$

and hence does not coincide with $\mathcal{M}$ (moreover, $\widetilde{\mathcal{N}}$ maps such pairs onto pairs of $\mathbb{H}(\mathbb{C})$-valued functions). Set

$$A_1 := \begin{pmatrix} \alpha & -\sigma \\ -\alpha & -\sigma \end{pmatrix}, \quad B_1 := \frac{1}{2}\begin{pmatrix} \sigma^{-1} & -\sigma^{-1} \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix},$$

then

$$A_1 * \mathcal{N} * B_1 = \begin{pmatrix} \overline{D}_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix},$$

where "$*$" stand for usual matrix multiplication.

Analogously for

$$A_2 := \begin{pmatrix} -\alpha & -\sigma \\ \alpha & -\sigma \end{pmatrix}, \quad B_2 := \frac{1}{2}\begin{pmatrix} -\sigma^{-1} & \sigma^{-1} \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix}$$

one has

$$A_2 * \mathcal{N} * B_2 = \begin{pmatrix} D_\alpha & 0 \\ 0 & \overline{D}_\alpha \end{pmatrix},$$

(all the matrices A_1, B_1, A_2, B_2 are invertible).

Thus there exist invertible matrices A_1, B_1, A_2, B_2 such that:

$$\mathcal{N} = A_1^{-1} * \begin{pmatrix} \overline{D}_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix} * B_1^{-1}, \quad \text{and} \quad \mathcal{N} = A_2^{-1} * \begin{pmatrix} D_\alpha & 0 \\ 0 & \overline{D}_\alpha \end{pmatrix} * B_2^{-1}.$$

This means, in particular, that

$$ker\mathcal{N} \approx kerD_\alpha \times ker\overline{D}_\alpha.$$

The "quaternionic Cauchy-Maxwell kernel", i.e., the fundamental solution of $\mathcal{N}$, is given by

$$\mathcal{K}_{\mathcal{N},\alpha}(x) := \frac{1}{2}\begin{pmatrix} \sigma^{-1} & \sigma^{-1} \\ \alpha^{-1} & -\alpha^{-1} \end{pmatrix} * \begin{pmatrix} \alpha\overline{\mathcal{K}}_\alpha(x) & -\sigma\overline{\mathcal{K}}_\alpha(x) \\ \alpha\mathcal{K}_\alpha(x) & \sigma\mathcal{K}_\alpha(x) \end{pmatrix},$$

where $\overline{\mathcal{K}}_\alpha$ is the Cauchy kernel for $\overline{D}_\alpha$.

The integral

$$K_{\mathcal{N},\alpha}[f](x) := -\int_\Gamma \mathcal{K}_{\mathcal{N},\alpha}(x-\tau) * \widetilde{\sigma}_\tau * f(\tau),\ x \in \Omega^\pm,$$

plays the role of the Cauchy-type integral, the one with the quaternionic Cauchy-Maxwell kernel (see [3], [8]); with $f : \Gamma \to Mat_{2\times 2}(\mathbb{H}(\mathbb{C}))$ and $\widetilde{\sigma}_\tau := \begin{pmatrix} 0 & \overline{\sigma}_\tau \\ \sigma_\tau & 0 \end{pmatrix}$. We shall call also $K_{\mathcal{N},\alpha}[f]$ the quaternionic Cauchy-Maxwell-type integral.

THEOREM 5.1 (Poincaré-Bertrand formula for the quaternionic Cauchy-Maxwell integral on a piece-wise Liapunov surface). Let Γ be a piece-wise Liapunov surface in $\mathbb{R}^3$. Let $f \in H_\mu(\Gamma \times \Gamma; Mat_{2\times 2}(\mathbb{H}(\mathbb{C}))),\ 0 < \mu < 1$, then the following formulas hold everywhere on Γ:

$$\int_{\Gamma_{\tau_1}}\int_{\Gamma_\tau} \mathcal{K}_{\mathcal{N},\alpha}(t-\tau) * \widetilde{\sigma}_\tau * \mathcal{K}_{\mathcal{N},\alpha}(\tau-\tau_1) * \widetilde{\sigma}_{\tau_1} * f(\tau,\tau_1) + \gamma^2(t)f(t,t)$$

$$= \int_{\Gamma_\tau}\int_{\Gamma_{\tau_1}} \mathcal{K}_{\mathcal{N},\alpha}(t-\tau) * \widetilde{\sigma}_\tau * \mathcal{K}_{\mathcal{N},\alpha}(\tau-\tau_1) * \widetilde{\sigma}_{\tau_1} * f(\tau,\tau_1). \tag{6}$$

PROOF. Let $f \in H_\mu(\Gamma \times \Gamma; Mat_{2\times 2}(\mathbb{H}(\mathbb{C})))$, consider $\mathcal{K}_{\mathcal{N},\alpha}$. Hence using formula (5) and after not complicated computation we obtain (6).

6 Proof of Theorem 2.1

In this Section we use results from Section 4. For the reader's convenience, recall some information from [8]: for $\vec{f} \in Mat_{2\times 1}(\mathbb{C}^3)$

$$\mathcal{K}_{\mathcal{N},\alpha}(\xi-\zeta) * \widetilde{\sigma}_\zeta * \vec{f}(\zeta) = \Big(\langle \mathcal{U}_\alpha(\xi-\zeta), \vec{f}(\zeta)\rangle_{Mat} + \mathcal{K}_{\mathcal{M}}(\xi,\zeta) \star \vec{f}(\zeta)\Big) ds(\zeta),$$

where $\langle\cdot,\cdot\rangle_{Mat}$ is defined (see [8, Section 4]). The proof of Theorem 2.1 follows from Theorem 5.1 taking into account the above relation between the class of the time-harmonic electromagnetic fields and α-hyperholomorphic functions.

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