Applied Mathematics E-Notes, 7(2007), 42-49 © ISSN 1607-2510
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Star Matrices: Properties And Conjectures*

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Received 18 February 2006

Abstract

Let Ω_n denote the set of all nxn doubly stochastic matrices. A matrix $B \in \Omega_n$ is said to be a star matrix if $\text{per}(\alpha B + (1-\alpha)A) \leq \alpha\ \text{per}B + (1-\alpha)\text{per}A$, for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$. In this paper we derive a necessary condition for a star matrix to be in Ω_n, and a partial proof of the star conjecture: The direct sum of two star matrices is a star matrix.

1 Introduction

Let Ω_n denote the set of all n by n doubly stochastic matrices. An interesting problem in the study of permanents is whether the permanent function is convex on Ω_n? That is, to see the validity of the inequality

$$\text{per}(\alpha B + (1-\alpha)A) \leq \alpha \text{per}B + (1-\alpha)\text{per}A, \tag{1}$$

for all $A, B \in \Omega_n$ and for all $\alpha \in [0,1]$. Though the result is true for $n = 2$, it is not true for $n \geq 3$. It was established by a counterexample given by Marcus and quoted by Perfect [5]. In view of the falsity of the convexity of the permanent function restricting B to some particular matrices in Ω_n, the validity of (1) for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$ was investigated by many authors. The first result on the convexity of permanent function obtained by Perfect [5], showed that $\text{per}\left(\frac{I_n+A}{2}\right) \leq \frac{1}{2} + \frac{1}{2}\text{per}A$. Brualdi and Newman [1] improved this result by showing that $\text{per}(\alpha I_n + (1-\alpha)A) \leq \alpha + (1-\alpha)\text{per}A$, for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$. Also they found that (1) is not valid for $B = J_3$ by considering $A = (3J_3 - I_3)/2$, but (1) holds for all $\alpha \in [\frac{1}{2}, 1]$, where J_n is a doubly stochastic matrix whose entries are $\frac{1}{n}$. Wang [6] called a matrix B in Ω_n a star, if B satisfies

$$\text{per}(\alpha B + (1-\alpha)A) \leq \alpha \text{per}B + (1-\alpha)\text{per}A, \tag{2}$$

for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$. A necessary condition for $B \in \Omega_n$ to be a star, $\text{per}B \geq 1/2^{n-1}$, is also found by Wang [6]. Brualdi and Newman[1] have derived a

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necessary and sufficient condition for $B \in \Omega_n$ to be a star, which states that, $B \in \Omega_n$ is a star if and only if

$$\sum_{i,j=1}^{n} b_{ij}\text{per}A_{ij} \leq \text{per}B + (n-1)\text{per}A \tag{3}$$

where A_{ij} is an $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and the j-th column of A. As $\alpha \in [0,1]$, inequality (3) is also written as,

$$\sum_{i,j=1}^{n} a_{ij}\text{per}B_{ij} \leq \text{per}A + (n-1)\text{per}B \tag{4}$$

It is easy to show that every matrix in Ω_2 is a star. For all n, I_n and P_n are stars, where P_n is the full cycle permutation matrix.

Karuppanchetty and Maria Arulraj [3] have disproved Wang's conjecture [6], which states that, for $n \geq 3$, permutation matrices are the only stars, by proving

$$B = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \oplus 1 \in \Omega_3,\ 0 \leq x \leq 1, \tag{5}$$

to be a star. They proved that this is the only star in Ω_3 up to permutations of rows and columns. They also established that the following are equivalent: (i) B is a star in Ω_n, (ii) B^T is a star and (iii) PBQ is a star for any two permutation matrices P and Q.

For brevity, let us use the notation $M(a,b;c,d)$ to denote the 3×3 doubly stochastic matrix

$$\begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{pmatrix}$$

and

$$E_1 = \begin{pmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{pmatrix},\ \varepsilon > 0.$$

The matrix $B = 1 \oplus M(a,b;c,d) \in \Omega_4$ where $0 < a,b < 1$ and $a+b \neq 1$, is not a star, since the only star in Ω_3 is $M(a,1-a;1-a,a)$ up to permutation of rows and columns.

For integers r and n, $(1 \leq r \leq n)$, let $Q_{r,n}$ denote the set of all sequences $(i_1, i_2, ..., i_r)$ such that $1 \leq i_1 < \ldots < i_r \leq n$. For fixed $\alpha, \beta \in Q_{r,n}$, let $A(\alpha/\beta)$ be a submatrix of A obtained by deleting the rows α and the columns β of A, let $A[\alpha/\beta]$ denote the submatrix of A formed by the rows α and the columns β of A and $T(A[\alpha/\beta])$ denotes the sum of all the elements of the matrix $A[\alpha/\beta]$. Let A_i denote the first n-3 columns of the i^{th} row of A and A^j denote the first n-3 rows of the j^{th} column of A. We denote $A+E$ as $\widetilde{A}$, a perturbation matrix of $A \in \Omega_n$.

In this paper, we frequently use the following results (Minc [4]): If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices, then

$$\text{per}A = \sum_{\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}A(\alpha/\beta), \quad \text{for} \quad \alpha \in Q_{r,n} \tag{6}$$

$$\sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}A(\alpha/\beta) = \left(\begin{array}{c} n \\ r \end{array} \right) \text{per}A, \tag{7}$$

and

$$\text{per}(A+B) = \sum_{r=0}^{n} S_r(A,B), \quad \text{where } S_r(A,B) = \sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}B(\alpha/\beta) \tag{8}$$

$\text{per}A[\alpha/\beta] = 1$ when $r = 0$ and per $A(\alpha/\beta) = 1$ when $r = n$.

2 Properties of Star Matrices

From the definition of star matrices, it is easy to verify that the average of two stars in Ω_2 is also a star in Ω_2. This is not so in Ω_n, for $n \geq 3$. For example, let $C = M\left(1, 0; 0, \frac{1}{2}\right)$ and $D = M\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right)$ be in Ω_3. Here C and D are stars, but $B = \frac{1}{2}(C+D) = M\left(\frac{3}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{2}\right)$ is not a star, since the matrix B defined by (5) is the only star in Ω_3 up to permutations of rows and columns. Hence the convex combination of two stars need not be a star in Ω_n, $n \geq 3$. The above example leads us to find a condition for the average of two stars to be a star in Ω_n.

THEOREM 1. Let C and D be stars in Ω_n. If per$C$+per$D$ $\leq$ 2 per B, then $B = \frac{1}{2}(C+D) \in \Omega_n$ is a star.

Indeed, let $A \in \Omega_n$. Then

$$\begin{aligned}
& \sum_{i,j=1}^{n} b_{ij}\text{per}A_{ij} - \text{per}B - (n-1)\text{per}A \\
= \; & \frac{1}{2}\left\{ \sum_{i,j=1}^{n} c_{ij}\text{per}A_{ij} + \sum_{i,j=1}^{n} d_{ij}\text{per}A_{ij} \right\} - \text{per}B - (n-1)\text{per}A \\
\leq \; & \frac{1}{2}\{\text{per}C + (n-1)\text{per}A + \text{per}D + (n-1)\text{per}A\} - \text{per}B - (n-1)\text{per}A \\
\leq \; & \frac{1}{2}\{\text{per}C + \text{per}D\} - \text{per}B \\
\leq \; & 0.
\end{aligned}$$

LEMMA 1. Let $B \in \Omega_n$. If there exists an $n \times n$ matrix $E \neq 0$, such that the perturbation matrix $\widetilde{B} = B + E \in \Omega_n$ and $\sum_{k=0}^{n-2}(n-(k+1))S_k(B,E) < 0$, then B is not a star.

Indeed, it is easy to show that,

$$\sum_{i,j=1}^{n} b_{ij} S_k(B_{ij}, E_{ij}) = (k+1)S_{k+1}(B,E),\ 0 \le k \le n-2,$$

and

$$\sum_{i,j=1}^{n} b_{ij} S_{n-1}(B_{ij}, E_{ij}) = \sum_{i,j=1}^{n} b_{ij} \mathrm{per} B_{ij}.$$

Let $A = \widetilde{B}$. Then

$$\begin{aligned}
& \sum_{i,j=1}^{n} b_{ij} \mathrm{per} \widetilde{B}_{ij} - \mathrm{per} B - (n-1)\mathrm{per}\widetilde{B} \\
= & \sum_{i,j=1}^{n} b_{ij} \sum_{k=0}^{n-1} S_k(B_{ij}, E_{ij}) - \mathrm{per} B - (n-1)\mathrm{per}\widetilde{B} \\
= & \sum_{k=1}^{n-1} k S_k(B,E) + \sum_{i,j=1}^{n} b_{ij} \mathrm{per} B_{ij} - \mathrm{per} B - (n-1)\mathrm{per}\widetilde{B} \\
= & \sum_{k=0}^{n-1} k S_k(B,E) + n\mathrm{per} B - \mathrm{per} B - (n-1)\sum_{k=0}^{n} S_k(B,E) \\
= & \sum_{k=0}^{n-1} k S_k(B,E) - (n-1)\sum_{k=0}^{n-1} S_k(B,E) \\
= & -\sum_{k=0}^{n-2} (n-(k+1)) S_k(B,E) \\
> & \ 0.
\end{aligned}$$

Let $B = (b_{ij}) \in \Omega_n$, permute B such that $b_{n-2,n}, b_{n-1,n-2}$ and $b_{n,n-1}$ are positive. Using Lemma 1, we have a necessary condition for the matrix B to be a star.

THEOREM 2. Let $B = (b_{ij})$ be in Ω_n such that $b_{n-2,n}, b_{n-1,n-2}$ and $b_{n,n-1}$ are positive. If B is a star, then

$$\left(\sum_{i,j=n-2}^{n} b_{ij} - 2 \sum_{i,j=n-2}^{n} b_{ii} \right) \mathrm{per} X + \sum_{i,j=1}^{n-3} \left(\sum_{k,r=n-2}^{n} b_{ri} b_{jk} - 2 \sum_{k=n-2}^{n} b_{ki} b_{jk} \right) \mathrm{per} X(j/i) \tag{9}$$

is nonnegative, where $X = (b_{ij})_{(n-3)\times(n-3)}$ is a submatrix of B formed by taking the first $n-3$ rows and $n-3$ columns of B.

PROOF: Let $E = \mathbf{0}_{(n-3)\times(n-3)} \oplus E_1$, $\mathbf{0}_{(n-3)\times(n-3)}$ is the zero matrix of order $n-3$, such that the perturbation matrix $\widetilde{B} = B + E$ is in Ω_n. Let us suppose that B is a star, then by Lemma 1, $\sum_{k=0}^{n-2}(n-(k+1))S_k(B,E) \ge 0$. It is easy to show that,

$S_k(B,E)=0$, for $k=0,...,n-3$. Now,

$$S_{n-2}(B,E)=\epsilon^2\left(\sum_{i,j=n-2,i\neq j}^{n}\text{per}\begin{pmatrix} X & B^j \\ B_i & b_{ij}\end{pmatrix}-\sum_{i=n-2}^{n}\text{per}\begin{pmatrix} X & B^i \\ B_i & b_{ii}\end{pmatrix}\right).$$

Take the permanent through the last row, we get

$$\begin{aligned} S_{n-2}(B,E) &= \epsilon^2\left(\sum_{i,j=n-2,i\neq j}^{n} b_{ij}-\sum_{i=n-2}^{n} b_{ii}\right) \\ &\quad +\epsilon^2\sum_{i=1}^{n-3}\sum_{j=n-2}^{n}\left[\sum_{r=n-2,j\neq r}^{n} b_{ri}\text{per}(X(i)B^j)-b_{ji}\text{per}(X(i)B^j)\right]\end{aligned}$$

where $(X(i)B^j)$ is a submatrix of order $n-3$ formed by deleting the i-th column of X and includes the column B^j. Now, taking permanent of $(X(i)B^j)$ through the column B^j we get

$$\begin{aligned} S_{n-2}(B,E) &= \epsilon^2\left(\sum_{i,j=n-2}^{n} b_{ij}-2\sum_{i=n-2}^{n} b_{ii}\right)\text{per}X \\ &\quad +\epsilon^2\sum_{i,j=1}^{n-3}(\sum_{k,r=n-2}^{n} b_{ri}b_{jk}-2\sum_{k=n-2}^{n} b_{ki}b_{jk})\text{per}X(j/i).\end{aligned}$$

Hence the necessary and sufficient condition for $B\in\Omega_n$ to be a star is that (9) is nonnegative.

Permute the identity matrix I_n such that the values in the positions $(n-2,n)$, $(n-1,n-2)$ and $(n,n-1)$ are one. Hence it is easy to very that I_n satisfies the condition of the Theorem 2.

COROLLARY 1. Let $B=(b_{ij})$ be in Ω_4 such that b_{24},b_{32} and b_{43} are positive. If B is a star, then

$$b_{kk}\left(\sum_{i=1,i\neq k}^{4} b_{ii}-b_{kk}\right)+\sum_{j=1,j\neq k}^{4} b_{kj}b_{jk}\leq\frac{1}{2},\ k=1,2,3,4.$$

PROOF. Without loss of generality, we prove this corollary for $k=1$. Let $E=0\oplus E_1,\ \epsilon>0$, such that the perturbation matrix $\widetilde{B}=B+E\in\Omega_4$. Let $X=(b_{11})$. From (8), per$X(1/1)=1$. Suppose B is a star, then the Theorem 2 becomes,

$$\begin{aligned} &\left(\sum_{i,j=n-2}^{n} b_{ij}-2\sum_{i=n-2}^{n} b_{ii}\right)\text{per}X+\sum_{i,j=1}^{n-3}\left(\sum_{k,r=n-2}^{n} b_{ri}b_{jk}-2\sum_{k=n-2}^{n} b_{ki}b_{jk}\right)\text{per}X(j/i) \\ =\ & b_{11}(3-(b_{21}+b_{31}+b_{41})-2(b_{22}+b_{33}+b_{44}))+b_{12}(1-b_{11}) \\ & +b_{13}(1-b_{11})+b_{14}(1-b_{11})-2(b_{12}b_{21}+b_{13}b_{31}+b_{14}b_{41}) \\ \geq\ & 0.\end{aligned}$$

That is,

$$(-2b_{11}(b_{22}+b_{33}+b_{44}-b_{11})+1-2(b_{12}b_{21}+b_{13}b_{31}+b_{14}b_{41}))\geq 0.$$

This implies that,

$$b_{11}(b_{22}+b_{33}+b_{44}-b_{11})+(b_{12}b_{21}+b_{13}b_{31}+b_{14}b_{41})\leq \frac{1}{2}.$$

The condition (9) in the Theorem 2 is only necessary but not sufficient. For example, J_n satisfies the condition (9) in the Theorem 2, but J_n is not a star.

3 Direct Sum of Star Matrices

It follows from the definition of star matrix that, for $n = 2$, every doubly stochastic matrix is a star. In general, since the permanent is invariant under permuting rows and columns and from the theorem of Brualdi and Newman [1] it follows that, all permutation matrices are stars. Wang [6] believed that for all $n \geq 3$, the only stars are permutation matrices and hence proposed a conjecture and quoted by Cheon and Wanless [2], which states that, "for $n \geq 3$, $B \in \Omega_n$ is a star if and only if B is a permutation matrix". Karuppanchetty and Maria Arulraj [3] have disproved Wang's conjecture, by proving the matrix B defined by (5) is a star. For disproving the conjecture in more general case, Karuppanchetty and Maria Arulraj [3] (also see Cheon and Wanless [2]) observed that, the stars in Ω_n are only direct sum of 2×2 doubly stochastic matrices and identity matrices. In this regard they proposed the following conjectures:

i. The direct sum of two stars is also a star.

ii. The only stars in Ω_n are the direct sum of 2×2 doubly stochastic matrices and identity matrices up to permutations of rows and columns.

In our endeavor to prove the first conjecture, we establish the conjecture only partially in the sense that the condition for star is satisfied for all A in Ω_n, by permuting A with specified conditions.

For example,

$$B=\begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}\oplus\begin{pmatrix} y & 1-y \\ 1-y & y \end{pmatrix}\in\Omega_4,\quad 0\leq x,\ y\leq 1, \qquad (10)$$

satisfies the star condition for the matrix $A = (a_{ij}) \in \Omega_4$ such that $a_{11}+a_{12}+a_{21}+a_{22} \leq 1$. This result is established in Theorem 3. The matrix

$$B=I_n\oplus\begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}\in\Omega_n,\ 4\leq n,\ 0\leq x\leq 1,$$

satisfies the star condition for all $A = (a_{ij}) \in \Omega_n$ such that a_{11} and $a_{22} \leq \frac{1}{n}$. This result is proved in Theorem 4.

THEOREM 3. The direct sum of two 2×2 doubly stochastic matrices satisfies the star condition for the matrices $A = (a_{ij}) \in \Omega_4$ such that $a_{11} + a_{12} + a_{21} + a_{22} \leq 1$.

PROOF. Let B be the matrix defined by (10). Without loss of generality let us assume that both x and y are at least $\frac{1}{2}$. Let $A = (a_{ij})$ be in Ω_4 such that $T(A[(1,2)/(1,2)]) \leq 1$. Now,

$$\begin{aligned}
&\sum_{i,j=1}^{4} a_{ij}\text{per}B_{ij} - \text{per}A - 3\text{per}B \\
= \;& (a_{11}+a_{22})x(2y^2-2y+1) + (a_{12}+a_{21})(1-x)(2y^2-2y+1) \\
&+(a_{33}+a_{44})y(2x^2-2x+1) + (a_{34}+a_{43})(1-y)(2x^2-2x+1) \\
&-\text{per}A - 3(2x^2-2x+1)(2y^2-2y+1) \\
\leq \;& T(A[(1,2)/(1,2)])x(2y^2-2y+1) + T(A((1,2)/(1,2)))y(2x^2-2x+1) \\
&-\text{per}A - 3(2x^2-2x+1)(2y^2-2y+1) \\
\leq \;& x(2y^2-2y+1) + y(2x^2-2x+1) - \text{per}A - 3(2x^2-2x+1)(2y^2-2y+1) \\
\leq \;& (2y^2-2y+1)\left(x - \frac{3}{2}(2x^2-2x+1)\right) \\
&+(2x^2-2x+1)\left(y - \frac{3}{2}(2y^2-2y+1)\right) - \text{per}A \\
\leq \;& 0,
\end{aligned}$$

where the second inequality follows from $T(A[(1,2)/(1,2)]) = T(A((1,2)/(1,2))) \leq 1$, while the third from $x - \frac{3}{2}(2x^2-2x+1) \leq 0$,and $y - \frac{3}{2}(2y^2-2y+1) \leq 0$.

THEOREM 4. The matrix

$$B = I_{n-2} \oplus \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \in \Omega_n,\; n \geq 4,\; 0 \leq x \leq 1,$$

satisfies the star condition for all $A = (a_{ij}) \in \Omega_n$ such that a_{11} and $a_{22} \leq \frac{1}{n}$.

PROOF. Let $A = (a_{ij})$ be in Ω_n such that a_{11} and $a_{22} \leq \frac{1}{n}$. Without loss of generality let us assume that x is at least $\frac{1}{2}$. Now,

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}\text{per}B_{ij} - \text{per}A - (n-1)\text{per}B \\
= \;& \left(\sum_{i=1}^{n-2} a_{ii}\right)(2x^2-2x+1) + (a_{n-1,n-1} - a_{nn})x + (a_{n-1,n} + a_{n,n-1})(1-x) - \text{per}A \\
&-(n-1)(2x^2-2x+1) \\
\leq \;& (2x^2-2x+1)\left(\frac{2}{n} + n - 4 - (n-1)) + 2x - \text{per}A\right) \\
\leq \;& -\frac{5}{2}(2x^2-2x+1) + 2x - \text{per}A \\
\leq \;& 0,
\end{aligned}$$

where the first inequality follows from $T(A[(n-1,n)/(n-1,n)]) \leq 2$, while the third from $2x - \frac{5}{2}(2x^2-2x+1) \leq 0$.

4 Conclusion

If A and B are in Ω_n, then AB and BA are also in Ω_n. Hence there is an open question, whether the product of two stars is a star? The answer is yes for $n = 2$, since any 2x2 doubly stochastic matrix is a star. In the case of $n = 3$, if $A = M(\frac{1}{2}, 0; 0, 1)$ and $B = M(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})$, then $AB = M(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}, \frac{1}{2})$ is not a star. But if $A = M(x, 1-x; 1-x, x)$ and $B = M(y, 1-y; 1-y, y)$, $0 < x, y < 1$, then $AB = M(z, 1-z; 1-z, z)$ is a star, where $z = xy + (1-x)(1-y)$. For $n \geq 4$, there is no definite answer. But for some particular cases this result is true. For example, if B is a star in Ω_n, then PB and BP are also stars in Ω_n, where P is a permutation matrix.

We feel that the conjecture (i) "the direct sum of two stars is also a star", cannot be proved in general cases, since for any arbitrary matrices $A_1 \in \Omega_{n_1}$ and $A_2 \in \Omega_{n_2}$, where $n = n_1 + n_2$, cannot be expressed in terms of an arbitrary matrix A in Ω_n. However, for particular cases we can prove this conjecture. In this connection the theorems 3 and 4 give a partial proof for the conjecture (i). To prove the conjecture (ii), there are two possible lines of attack. One could take a positive matrix and prove that it is not a star and the other way is, any doubly stochastic matrix with an odd number of zeros is not a star. In this regard, we conclude this paper by proposing the following conjectures.

Conjecture (1): Any positive matrix in Ω_n, $n \geq 4$, is not a star.

Conjecture (2): If B is a star in Ω_n, $n \geq 4$, then B is a symmetric matrix up to permutations of rows and columns.

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