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Orders With Ten Elements Are Circle Order*

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Abstract

With a new method based on the notion of genetic algorithm and the explicit enumeration of orders, we prove that all orders on at most 10 elements are circle orders. This theorem represents the best partial result on Sidney-Sidney-Urrutia Conjecture.

1 Order Dimension and Circle Orders

We are interested in orders of small size which are circle orders, in relation to their dimension (see [6]). As is well known, the finite posets of dimension at most two are just those which have inclusion representations using closed intervals of the real line $\mathbb{R}$. Because a closed interval of $\mathbb{R}$ can also be considered as a sphere in $\mathbb{R}^1$, it is natural to ask which posets have inclusion representations using circular disks in $\mathbb{R}^2$. For historical reasons, these posets are called *circle orders*. Schneinerman and Wierman [11] showed in that $\mathbb{Z}^3$ is not a circle order, and then Hulbert [10] showed that the same holds for $\mathbb{N}^3$. Urrutia, after having proved that all finite orders with dimension at most three are regular n-gon orders for all $n \geq 3$, conjectured that all finite orders with dimension at most three are circle orders (see [13]). However, in 1999, Felsner, Fishburn and Trotter [5] disproved this conjecture by using a new argument from Ramsey theory. They think it would be possible to find an order which is not a circle order with dimension three and at most one hundred elements. The *crossing number* of an order is the smallest integer m such that the order can be represented by continuous real-valued functions in $[0, 1]$, no two of whose graphs intersect in more than m points. With this notion, Sidney, Sidney and Urrutia [12] showed the following result.

THEOREM 1 [Sidney, Sidney and Urrutia, 1988]. For $n \geq 4$, Ψ_n is not a circle order.

To construct the family Ψ_n, start with the standard n-dimension order denoted by $H_n(Y, <)$, where $Y = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, and $u_i < v_j$ for $i \neq j$, the other elements

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being incomparable. Then for every subset S_k of $\{1,\ldots,n\}$ with exactly $\lfloor\frac{n}{2}\rfloor$ or $\lfloor\frac{n+1}{2}\rfloor$ elements (if n is even, both values are the same, if n is odd they are different), insert in $H_n(Y,<)$ a new element s_k such that $s_k > u_j$ for $j \in S_k$, $s_k < v_i$ for $i \notin S_k$ and finally, $s_k > s'_k$ if $S'_k \subset S_k$.

To show that the dimension of Ψ_n is exactly n, it is sufficient to observe that Ψ_n is contained in 2^n and contains H_n.

Furthermore in 1989, thanks to Radon lemma, Brightwell and Winkler [1] obtained the following theorem.

THEOREM 2 [Brightwell and Winkler, 1989]. For each $n \geq 1$, the order with $2^{n+2}-2$ elements $P_{2^{n+2}-2}$ is a n-sphere order, but not a $(n-1)$-sphere order, where

$$P_{(2^{n+2}-2)} = \{A \subseteq \{1,2,\ldots,n+2\} \,:\, 1 \leq |A| \leq n+1\}$$

with $A \leq B$ if and only if $A \subseteq B$ and ($|A| = 1$ or $|B| = n+1$).

Among those two families of orders which are not circle orders, there is an order with 14 elements belonging to both families: $P_2 = \Psi_4$, we call it p_{14}.

CONJECTURE 1 [Sidney, Sidney and Urrutia, 1988]. The smallest order which is not a circle order is p_{14}.

In 1991, Fraïssé and Lygeros [7] showed that all orders with at most 7 elements are circle orders, introducing a method based upon an exhaustive order computation and their automatic discrimination with the application of the theorem of Dushnik and Miller [4], and Hiraguchi [9]. However, the last part of the proof which is a fastidious case by case verification, cannot be applied to larger orders. So, it is necessary to introduce a new method.

2 Enumeration and Genetic Algorithm

We consider the list of non-isomorphic orders with n elements, obtained by the generation algorithm of Chaunier and Lygeros (see [2, 3]). Every order is represented in a canonical way by a list of $\frac{n(n-1)}{2}$ bits. If $P = (X,<)$ is a poset then for $(x,y) \in X^2$ we write $x \parallel y$ if and only if x and y are incomparable for $<$. A *conjugate* of a poset $P = (X,<)$ is a poset $P' = (X,\prec)$ such that $\forall (x,y) \in X^2$, $x \parallel y$ if and only if ($x \prec y$ or $y \prec x$). As every order with dimension at most two is clearly a circle order, we use the following theorem to detect them (see [4]).

THEOREM 3 [Dushnik and Miller, 1941]. A poset has dimension at most two if and only if it has a conjugate order.

Thus, to know whether or not the dimension of a poset P is at most two, we only need to know whether or not the graph $G = (X, \{(x,y) \in X^2, x \parallel y\})$ — called the *incomparability graph* of the poset P — admits a transitive orientation, which can be done in linear time (see, for instance, [8]).

We obtain the following proportions.

n	Proportion of 2-dimensional orders among orders with n elements
7	95.65% that is 1956/2045
8	87.03% that is 14794/16999
9	71.78% that is 131526/183231
10	51.88% that is 1331848/2567284

We explicit now some basic properties of a minimal counter-example $P(X,<)$ to the conjecture, regarding the number of elements. Let n be the number of elements of P — therefore, all orders with $n-1$ elements are circle orders. A *sink* is an element $x \in X$ such that $\forall y \in X \setminus \{x\}, y < x$. It is clear that P cannot have a sink. A *source* is an element $x \in X$ such that $\forall y \in X \setminus \{x\}, x < y$. Then, P has no source. This is by virtue of the existence of Sidney-Sidney-Urrutia's *normal representation*: be it by increasing the circles radius, every circle order admits a part of the plane that is shared by all the circles. Besides, we also know that P does not have two twin elements, where $(x,y) \in X^2$ are *twin elements* if and only if

- $x < y$;
- $\forall z \neq y, x < z \Rightarrow y < z$; and
- $\forall z \neq x, z < y \Rightarrow z < x$.

There is a natural notion of duality for posets: let $(X,<)$ be a poset, the binary relation $\prec$ defined on X by

$$\forall (x,y) \in X^2, x \prec y \text{ if and only if } y < x$$

is a partial order. The poset $(X,\prec)$ is called the *dual* of the poset $(X,<)$. As is well known (see [13]), the dual of a finite circle order is also a circle order.

Let $n \in \{8,9,10\}$, and consider the enumeration of all non-isomorphic partial orders with n elements. We remove from the list every order that has a sink, a source or two twin elements. Moreover, we remove an order in each couple of dual orders. It remains 783 orders with 8 elements, 16537 orders with 9 elements and 397603 orders with 10 elements, on which we run the representation search algorithm.

This program comes from genetic algorithms. To find a representation of a given order P with n elements, we start by randomly generating a population of subjects: a subject is a set of n circles, fulfilling some inclusion constraints. Every subject is created in a random way, the circles representing elements of big degree being more likely to have smaller radii. Then, we define the fitness of every subject: the bigger it is, the more the subject is close to a representation of P. A subject whose fitness reaches the maximum value is a solution. To compute the fitness of a subject, we apply a bonus of value $n+1$ for each relation present in the subject that also is present in P. In the same way, a bonus of one is given for any two incomparable elements of the subject that also are incomparable in P. On the contrary, if a relation is missing in the subject, then a penalty of value $2n$ is applied. For the relations present in the subject but not in the original poset, we distinguish two cases: consider the height of the elements in the Hasse diagram of P, where minima of P have height zero while maxima have maximum height. Let $x < y$ be a relation present in the subject but not in P. If the height of x is less than the height of y, then the value of the penalty applied is n. But if the height of x is at least the height of y then the value of the penalty applied is $2n$. Note that this test is elementary thanks to the canonical representation of the poset P given by the enumeration algorithm. Thus the fitness of any subject is at most $\frac{n(n-1)}{2} + n \times$ (number of relations of P), with equality if and only if the subject is a representation of P.

The following three steps form a so-called *generation*. A maximum number of generations is fixed at the beginning. If the algorithm has not found a representation after this number of generations, it stops.

1. Each subject, except the best two, undergoes *local disturbances* so as to improve its fitness;

2. the fitness of each subject is re-evaluated; and

3. worst subjects are replaced by new ones without any crossing process.

We have implemented two kinds of local disturbances: the algorithm can remove an inclusion between two circles, or force an inclusion.
To remove the inclusion of $\mathcal{C}_1$ in $\mathcal{C}_2$, we move the circle that scores less points in the fitness evaluation process of the subject. The centre is translated in a random direction, with the minimum ratio that cancels the inclusion.
To force the inclusion of $\mathcal{C}_1$ in $\mathcal{C}_2$, again only the circle that scores less points in the fitness evaluation process of the subject is changed, say $\mathcal{C}_1$. The forcing aims at minimising the changes of position. The radius of $\mathcal{C}_1$ becomes $r \times$ (radius of $\mathcal{C}_2$) where r is randomly chosen between $\frac{1}{2}$ and $\frac{3}{4}$. Then the centre of $\mathcal{C}_1$ is translated in the direction defined by the two centres, with a ratio that minimises the movement and achieves the inclusion.

Between two generations, some subjects of the population might undergo some mutations. The subjects are chosen randomly, and a mutation consists of forcing some needed inclusion not achieved by the subject.

All the solutions returned by the representation search algorithm were validated by a formal certificate. This certificate computes and compares only integers, using the GMP multiprecision library.

With this new method, we extended the previous result on orders with at most 7 elements to orders with up to 10 elements. We thus obtained the following theorem.

THEOREM 4. All orders with at most 10 elements are circle orders.

We underline here the importance of the local aspect of the disturbances applied to the subjects. For instance, when moving a circle $\mathcal{C}$ so as to force or to remove an inclusion, if we move accordingly all the circles that $\mathcal{C}$ contains, or that are contained by $\mathcal{C}$, then the algorithm is much less powerful.

As the Urrutia conjecture on 3-dimensional orders has been disproved by the result of [5], it does not exist for the moment a global method able to directly prove Sidney-Sidney-Urrutia conjecture. Thus our method, even if it is exhaustive, remains the only one which can theoretically prove it. Furthermore, the representation search algorithm has always found a representation until now. A failure of the algorithm could exhibit a potentially good candidate to disprove the conjecture. Hence our method can also help in finding a counter-example, and by the way increase our understanding of circle orders.

3 Numerical Tables

These tables show the number of orders that remain after the discrimination process.

Orders with 8 elements

relations	orders	relations	orders
7	3	14	123
8	13	15	96
9	28	16	78
10	59	17	38
11	82	18	21
12	115	19	4
13	121	20	2

Orders with 9 elements

relations	orders	relations	orders
8	11	19	1735
9	43	20	1381
10	136	21	959
11	303	22	612
12	593	23	334
13	932	24	153
14	1374	25	62
15	1721	26	22
16	2022	27	4
17	2111	28	2
18	2027		

Orders with 10 elements

relations	orders	relations	orders
9	37	23	35719
10	166	24	30697
11	554	25	24492
12	1483	26	18349
13	3231	27	12678
14	6157	28	8212
15	10351	29	4811
16	15795	30	2642
17	22047	31	1257
18	28517	32	571
19	34189	33	214
20	38273	34	76
21	39947	35	23
22	39111	36	7

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