**Applied Mathematics E-Notes, 7(2007), 154-158 ©** ISSN 1607-2510
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# Global Attraction In A Rational Recursive Relation*

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Received 20 May 2007

**Abstract**

A real sequence $\{y_i\}_{i=-m}^{\infty}$ defined by any $y_{-m}, y_{-m+1}, ..., y_{-1} \in (0, \infty)$ and the rational recursive relation (1) will converge to 1.

In [1], a question is raised as to whether a real sequence $\{y_i\}_{i=-m}^{\infty}$ that satisfies (any) $y_{-m}$, $y_{-m+1}$, ..., $y_{-1} \in (0, \infty)$ and

$$y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1}, \quad n \in N_0 = \{0, 1, 2, ...\}, \tag{1}$$

where $k, l, m$ are positive integers such that $1 \le k < l < m$, will converge to 1. Two different affirmative proofs are given in [2] and [3]. In this note, we offer another simple proof based on analysis of properties of subsequences of solutions of (1) (see e.g. [4] for another demonstration of such a technique). Since no unifying theory is available for rational recursive relations yet, such an addition may be of interest in future developments.

Consider a slightly more general rational recursive relation

$$y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m} + a}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1 + a}, \ n \in N_0 \tag{2}$$

where $a \ge 0$ and $k, l, m$ are positive integers such that $1 \le k < l < m$. Given $y_{-m}, y_{-m+1}, ..., y_{-1} \in (0, \infty)$, we may calculate $y_0, y_1, ...$ from (2) in a unique manner. The resulting positive sequence $\{y_n\}_{n=-m}^{\infty}$ will be called a solution of (2). For instance, the constant sequence $\overline{y} = \{1\}_{n=-m}^{\infty}$ is a solution (which is easily seen to be the unique positive equilibrium solution of (2)). Given a solution $\{y_n\}_{n=-m}^{\infty}$, the (positive) subsequences $\{y_{tm+i}\}_{t=-1}^{\infty}$, $i = 0, 1, ..., m-1$, will be denoted by $\Psi^{(i)}$, $i = 0, 1, ..., m-1$, respectively.

*Mathematics Subject Classifications: 40A05

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We first make the observation that if $\Psi_{t-1}^{(0)} = 1$ for some $t \geq 0$, then $\Psi_s^{(0)} = 1$ for $s \geq t$. Indeed,

$$\begin{aligned}\Psi_t^{(0)} &= \frac{y_{tm-k}y_{tm-l}\Psi_{t-1}^{(0)} + y_{tm-k} + y_{tm-l} + \Psi_{t-1}^{(0)} + a}{y_{tm-k}y_{tm-l} + y_{tm-k}\Psi_{t-1}^{(0)} + y_{tm-l}\Psi_{t-1}^{(0)} + 1 + a} \\ &= \frac{y_{tm-k}y_{tm-l} + y_{tm-k} + y_{tm-l} + 1 + a}{y_{tm-k}y_{tm-l} + y_{tm-k} + y_{tm-l} + 1 + a} \\ &= 1,\end{aligned}$$

and by induction, $\Psi_s^{(0)} = 1$ for $s \geq t+1$. The same reason is valid for the other sequences $\Psi^{(i)}$.

LEMMA 1. For each $i \in \{0, ..., m-1\}$, if $\Psi_{t-1}^{(i)} = 1$ for some $t \geq 0$, then $\Psi_n^{(i)} = 1$ for $n \geq t$.

Before stating the next result, recall that two real sequences $\{a_n\}$ and $\{b_n\}$ are said to be asymptotically equal if $a_n = b_n$ for all large $n$.

LEMMA 2. Let $\{y_n\}_{n=-m}^{\infty}$ be a solution of (2) and $\delta = \min\{y_{-m},\ y_{-m+1},\ \dots, y_{-1}\}$. Then, for any $\varepsilon \in (0, \delta)$, there exist positive integers $M_0, ..., M_{m-1}$ and positive numbers $\Phi_0, ..., \Phi_{m-1}$ such that

$$n > M_i \Rightarrow \left|\Psi_n^{(i)} - \Phi_i\right| < \varepsilon \text{ or } \left|\Psi_n^{(i)} - \frac{1}{\Phi_i}\right| < \varepsilon \tag{3}$$

for $i = 0, ..., m-1$.

PROOF. We will assume that $i = 0$, since the other cases are similar. If $\Psi^{(0)}$ is asymptotically equal to $\{1\}$, then our assertion is true. Suppose $\Psi^{(0)}$ is not asymptotically equal to $\{1\}$. Then by Lemma 1, the sequence $\Psi_n^{(0)} \neq 1$ for all large $n$. Hence the sequence $\Psi^{(0)} - \overline{y}$ is either eventually positive, or eventually negative, or oscillatory (i.e. neither eventually positive nor eventually negative). In the first case, we may assume without loss of generality that $\Psi_t^{(0)} > 1$ for all $t \geq -1$. A direct calculation then shows that

$$y_n - y_{n-m} = \frac{(1 - y_{n-m})\left[(1 + y_{n-m})(y_{n-k} + y_{n-l}) + a\right]}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1 + a}, \quad n \geq 0,$$

from which we get

$$(y_n - y_{n-m})(y_{n-m} - 1) < 0, \quad n \geq 0. \tag{4}$$

Hence

$$\Psi_{-1}^{(0)} > \Psi_0^{(0)} > \Psi_1^{(0)} > \cdots > 1,$$

which shows that $\Psi^{(0)}$ is a decreasing sequence bounded below by 1. If we take $\Phi_0 = \lim_{t\to\infty} \Psi_t^{(0)}$, then (3) is true for $i = 0$.

Similarly, we may show that in the second case, $\Psi^{(0)}$ is an (eventually) increasing sequence bounded above by 1 and (3) is true by taking $\Phi_0 = \lim_{t\to\infty} \Psi_t^{(0)}$.

In the case where $\Psi^{(0)} - \overline{y}$ is oscillatory, we may assume without loss of generality that $\Psi_{-1}^{(0)} > 1$. Then we may build an integer sequence $\{s_1, s_2, s_3, ...\}$ where $s_1$ denotes the number of first consecutive positive terms of $\Psi^{(0)} - \overline{y}$, $s_2$ is number of first consecutive negative terms of $\Psi^{(0)} - \overline{y}$, etc. In view of (4), we may then see that

$$\Psi^{(0)}_{-1} > \Psi^{(0)}_{0} > ... > \Psi^{(0)}_{s_1-2} > 1, \tag{5}$$

$$\Psi^{(0)}_{s_1-1} < \Psi^{(0)}_{s_1} < ... < \Psi^{(0)}_{s_1+s_2-2} < 1, \tag{6}$$

and inductively,

$$\Psi^{(0)}_{s_1+s_2+\cdots+s_p-1} > \Psi^{(0)}_{s_1+s_2+\cdots+s_p} > \cdots > \Psi^{(0)}_{s_1+s_2+\cdots+s_p+s_{p+1}-2} > 1,$$

$$\Psi^{(0)}_{s_1+s_2+\cdots+s_{p+1}-1} < \Psi^{(0)}_{s_1+s_2+\cdots+s_{p+1}} < \cdots < \Psi^{(0)}_{s_1+s_2+\cdots+s_{p+2}-2} < 1,$$

for $p \geq 1$. In view of (2) and (5),

$$\begin{aligned}
&\Psi^{(0)}_{s_1-1} \\
=\ & \frac{y_{-m+s_1m-k}y_{-m+s_1m-l}\Psi^{(0)}_{s_1-2} + y_{-m+s_1m-k} + y_{-m+s_1m-l} + \Psi^{(0)}_{s_1-2} + a}{y_{-m+s_1m-k}y_{-m+s_1m-l} + y_{-m+s_1m-k}\Psi^{(0)}_{s_1-2} + y_{-m+s_1m-l}\Psi^{(0)}_{s_1-2} + 1 + a} \\
>\ & \frac{y_{-m+s_1m-k}y_{-m+s_1m-l}\Psi^{(0)}_{s_1-2} + y_{-m+s_1m-k} + y_{-m+s_1m-l} + \Psi^{(0)}_{s_1-2} + a}{\Psi^{(0)}_{s_1-2}\left(y_{-m+s_1m-k}y_{-m+s_1m-l}\Psi^{(0)}_{s_1-2} + y_{-m+s_1m-k} + y_{-m+s_1m-l} + \Psi^{(0)}_{s_1-2} + a\right)} \\
=\ & \frac{1}{\Psi^{(0)}_{s_1-2}}.
\end{aligned} \tag{7}$$

Similarly, by (2) and (6),

$$\Psi^{(0)}_{s_1+s_2-1} < \frac{1}{\Psi^{(0)}_{s_1+s_2-2}}. \tag{8}$$

By induction, we may then show that

$$\Psi^{(0)}_{s_1+s_2+\cdots+s_{2q+1}-1}\Psi^{(0)}_{s_1+s_2+\cdots+s_{2q+1}-2} > 1,$$

and

$$\Psi^{(0)}_{s_1+s_2+\cdots+s_{2q+2}-1}\Psi^{(0)}_{s_1+s_2+\cdots+s_{2q+2}-2} < 1,$$

for $q \geq 0$. As a consequence,

$$\begin{aligned}
\Psi^{(0)}_{-1} &> \Psi^{(0)}_{0} > \cdots > \Psi^{(0)}_{s_1-2} \\
&> \frac{1}{\Psi^{(0)}_{s_1-1}} > \frac{1}{\Psi^{(0)}_{s_1}} > \cdots > \frac{1}{\Psi^{(0)}_{s_1+s_2-2}} \\
&> \cdots \\
&> \Psi^{(0)}_{s_1+s_2+\cdots+s_{2r}-1} > \Psi^{(0)}_{s_1+s_2+\cdots+s_{2r}} > \cdots > \Psi^{(0)}_{s_1+s_2+\cdots s_{2r+1}-2} \\
&> \frac{1}{\Psi^{(0)}_{s_1+s_2+\cdots s_{2r+1}-1}} > \frac{1}{\Psi^{(0)}_{s_1+s_2+\cdots s_{2r+1}}} > \cdots > \frac{1}{\Psi^{(0)}_{s_1+s_2+\cdots s_{2r+2}-2}} \\
&> \cdots
\end{aligned}$$

which shows that

$$\Psi^{(0)}_{s_1+s_2+\cdots+s_{2r}-1} > \frac{1}{\Psi^{(0)}_{s_1+s_2+\cdots+s_{2r+1}-1}} > \Psi^{(0)}_{s_1+s_2+\cdots+s_{2r+2}-1} \tag{9}$$

for $r \geq 1$ and that $\Psi^{(0)}$, when restricted to the positive support of $\Psi^{(0)} - \overline{y}$ is decreasing and bounded below by 1, and when restricted to the negative support of $\Psi^{(0)} - \overline{y}$ is increasing and bounded above by 1. Let $\Phi_0'$ and $\Phi_0''$ be the limit of $\Psi^{(0)}$ restricted to the positive and respectively the negative support of $\Psi^{(0)} - \overline{y}$. Then in view of (9), we see by taking limits that $\Phi_0' \geq \frac{1}{\Phi_0''} \geq \Phi_0'$, which shows $\Phi_0' = \frac{1}{\Phi_0''}$ as required. The proof is complete.

LEMMA 3. Let $\{y_n\}_{n=-m}^{\infty}$ be a solution of Eq. (2). Then, $\{y_n\}_{n=-m}^{\infty}$ converges to 1.

PROOF. It suffices to show that for each $i \in \{0, 1, ..., m-1\}$, $\Psi^{(i)}$ converges to 1. We will prove that $\Psi^{(0)}$ tends to 1, since the other $\Psi^{(i)}$ can be shown to converge to 1 in similar manners. By Lemma 2, for any $\varepsilon \in (0, \delta)$, where $\delta = \min\{y_{-m},\ y_{-m+1},\ \ldots, y_{-1}\}$, there are positive integers $M_0, ..., M_{m-1}$ and positive numbers $\Phi_0, ..., \Phi_{m-1}$ such that for $n > \{M_0,\ M_1,\ \ldots, M_{m-1}\} + m$, the following statements hold:

$$\left|\Psi^{(0)}_{n-1} - \Phi_0\right| < \varepsilon \text{ or } \left|\Psi^{(0)}_{n-1} - \frac{1}{\Phi_0}\right| < \varepsilon,$$

$$\left|\Psi^{(0)}_{n} - \Phi_0\right| < \varepsilon \text{ or } \left|\Psi^{(0)}_{n} - \frac{1}{\Phi_0}\right| < \varepsilon,$$

$$\left|\Psi^{(m-k)}_{n-1} - \Phi_{m-k}\right| = |y_{nm-k} - \Phi_{m-k}| < \varepsilon \text{ or } \left|\Psi^{(m-k)}_{n-1} - \frac{1}{\Phi_{m-k}}\right| < \varepsilon,$$

and

$$\left|\Psi^{(m-l)}_{n-1} - \Phi_{n-l}\right| < \varepsilon \text{ or } \left|\Psi^{(m-l)}_{n-1} - \frac{1}{\Phi_{m-l}}\right| < \varepsilon.$$

Consider the case where $\left|\Psi^{(0)}_{n-1} - \Phi_0\right| < \varepsilon$, $\left|\Psi^{(0)}_{n} - \Phi_0\right| < \varepsilon$, $\left|\Psi^{(m-k)}_{n-1} - \Phi_{m-k}\right| < \varepsilon$ and $\left|\Psi^{(m-l)}_{n-1} - \Phi_{n-l}\right| < \varepsilon$ hold. Then in view of (2),

$$\Psi^{(0)}_{n} = \frac{\Psi^{(m-k)}_{n-1}\Psi^{(m-l)}_{n-1}\Psi^{(0)}_{n-1} + \Psi^{(m-k)}_{n-1} + \Psi^{(m-l)}_{n-1} + \Psi^{(0)}_{n-1} + a}{\Psi^{(m-k)}_{n-1}\Psi^{(m-l)}_{n-1} + \Psi^{(m-k)}_{n-1}\Psi^{(0)}_{n-1} + \Psi^{(m-l)}_{n-1}\Psi^{(0)}_{n} + 1 + a}$$

so that

$$\begin{aligned}
& \Phi_0 - \varepsilon \\
< \ & \Psi^{(0)}_{n} \\
< \ & \frac{(\Phi_{m-k} + \varepsilon)(\Phi_{m-l} + \varepsilon)(\Phi_0 + \varepsilon) + (\Phi_{m-k} + \varepsilon) + (\Phi_{m-l} + \varepsilon) + (\Phi_0 + \varepsilon) + a}{(\Phi_{m-k} - \varepsilon)(\Phi_{m-l} - \varepsilon) + (\Phi_{m-k} - \varepsilon)(\Phi_0 - \varepsilon) + (\Phi_{m-l} - \varepsilon)(\Phi_0 - \varepsilon) + 1 + a}
\end{aligned}$$

and

$$\begin{aligned} & \Phi_0 + \varepsilon \\ > \; & \Psi_n^{(0)} \\ > \; & \frac{(\Phi_{m-k} - \varepsilon)(\Phi_{m-l} - \varepsilon)(\Phi_0 - \varepsilon) + (\Phi_{m-k} - \varepsilon) + (\Phi_{m-l} - \varepsilon) + (\Phi_0 - \varepsilon) + a}{(\Phi_{m-k} + \varepsilon)(\Phi_{m-l} + \varepsilon) + (\Phi_{m-k} + \varepsilon)(\Phi_0 + \varepsilon) + (\Phi_{m-l} + \varepsilon)(\Phi_0 + \varepsilon) + 1 + a}. \end{aligned}$$

By taking limits as $\varepsilon \to 0$ on both sides of the above two chain of inequalities, we see that

$$\Phi_0 = \frac{\Phi_{m-k}\Phi_{m-l}\Phi_0 + \Phi_{m-k} + \Phi_{m-l} + \Phi_0 + a}{\Phi_{m-k}\Phi_{m-l} + \Phi_{m-k}\Phi_0 + \Phi_{m-l}\Phi_0 + 1 + a}$$

or

$$(\Phi_0 - 1)\{(\Phi_0 + 1)(\Phi_{m-k} + \Phi_{m-l}) + a\} = 0.$$

Since $\Phi_0, \Phi_{m-k}, \Phi_{m-l}, a > 0$, we see that $\Phi_0 = 1$. The other cases can be handled in similar manners to yield $\Phi_0 = 1$. The proof is complete.

Lemma 3 can be rephrased as follows:

THEOREM 1. A real sequence $\{y_i\}_{i=-m}^{\infty}$ defined by any $y_{-m}, y_{-m+1}, ..., y_{-1} \in (0, \infty)$ and the rational recursive relation (2) will converge to 1.

**Acknowledgment.** This work was supported by the Doctor's Foundation of Lanzhou University of Technology.

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