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Existence Of Solutions Of Integrodifferential Evolution Equations With Time Varying Delays*

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Abstract

In this paper we prove the existence of mild solutions of nonlinear integrodifferential equations with time varying delays in Banach spaces. The results are obtained by using the resolvent operator and the Schaefer fixed point theorem. An application is provided to illustrate the technique.

1 Introduction

Using the method of semigroup, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations have been discussed by Pazy [11] and the nonlocal Cauchy problem for the same equation has been studied by Byszewskii [3, 4]. Balachandran and Chandrasekaran [1] studied the nonlocal Cauchy problem for semilinear integrodifferential equation with deviating argument. Balachandran and Park [2] has been discussed about the existence of solutions and controllability of nonlinear integrodifferential systems in Banach spaces. Grimmer [6] obtained the representation of solutions of integrodifferential equations by using resolvent operators in a Banach space. Liu [8] discussed the Cauchy problem for integrodifferential evolution equations in abstract spaces and also in [9] he discussed nonautonomous integrodifferential equations. Lin and Liu [7] studied the nonlocal Cauchy problem for semilinear integrodifferential equations by using resolvent operators. Liu and Ezzinbi [10] investigated non-autonomous integrodifferential equations with nonlocal conditions. Byszewskii and Acka [5] studied the classical solution of nonlinear functional differential equation with time varying delays. The purpose of this paper is to prove the existence of mild solutions for time varying delay integrodifferential evolution equations with the help of Schaefer's fixed point theorem.

2 Preliminaries

Consider the nonlinear time varying delay integrodifferential evolution equation of the form

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds + f(t, x(\sigma_1(t)), \int_0^t k(t,s,x(\sigma_2(s)))ds), \;\; t \in J \quad (1)$$

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with nonlocal condition

$$x(0) + g(x) = x_0, \tag{2}$$

where $A(t)$ and $B(t,s)$ are closed linear operators on a Banach space X with dense domain $D(A)$ which is independent of t, $f : J \times X \times X \to X$, $k : J \times J \times X \to X$, $g : C(J, X) \to X$ and the delay $\sigma_i(t) \leq t$ are given functions. Here $J = [0, T]$.

We shall make the following conditions:

(H_1) $A(t)$ generates a strongly continuous semigroup of evolution operators.

(H_2) Suppose Y is a Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $B(t,s)$ are closed operators it follows that $A(t)$ and $B(t,s)$ are in the set of bounded linear operators from Y to X, $B(Y,X)$, for $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively. $A(t)$ and $B(t,s)$ are continuous on $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively, into $B(Y,X)$.

DEFINITION 2.1. A resolvent operator for (1)-(2) is a bounded operator valued function $R(t,s) \in B(X)$, $0 \leq s \leq t \leq T$, the space of bounded linear operators on X, having the following properties

(i) $R(t,s)$ is strongly continuous in s and t. $R(t,t) = I$, the identity operator on X. $\|R(t,s)\| \leq Me^{\beta(t-s)}$ $t, s \in J$ and M, β are constants.

(ii) $R(t,s)Y \subset Y$, $R(t,s)$ is strongly continuous in s and t on Y.

(iii) For $y \in Y$, $R(t,s)y$ is continuously differentiable in s and t, and for $0 \leq s \leq t \leq T$,

$$\begin{aligned} \frac{\partial}{\partial t} R(t,s)y &= A(t)R(t,s)y + \int_s^t B(t,r)R(r,s)y dr, \\ \frac{\partial}{\partial s} R(t,s)y &= -R(t,s)A(s)y - \int_s^t R(t,r)B(r,s)y dr, \end{aligned}$$

with $\frac{\partial}{\partial t}R(t,s)y$ and $\frac{\partial}{\partial s}R(t,s)y$ are strongly continuous on $0 \leq s \leq t \leq T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in Banach spaces.

DEFINITION 2.2. A continuous function $x(t)$ is said to be a mild solution of the nonlocal Cauchy problem (1)-(2), if

$$x(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)f(s, x(\sigma_1(s)), \int_0^s k(s, \tau, x(\sigma_2(\tau)))d\tau)ds$$

is satisfied.

We need the following fixed point theorem due to Schaefer [12].

THEOREM 2.1. Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta(F) = \{x \in E : x = \lambda F x \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that the following conditions hold:

(H_3) There exists a resolvent operator $R(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M_1 > 0$ such that
$$\|R(t,s)\| \leq M_1.$$

(H_4) For each $t \in J$, the function $f(t,\cdot,\cdot) : X \times X \to X$ is continuous, and for each, $x \in X$ and the function $f(\cdot, x(\sigma_1(t)), \int_0^t k(t,s,x(\sigma_2(s)))ds) : J \to X$ is strongly measurable.

(H_5) There exists an integrable function $m_1 : J \times J \to [0,\infty)$ such that
$$\|k(t,s,x)\| \leq m_1(t,s)\Omega_0(\|x\|), \text{ for any } t,s \in J,\ x \in X,$$
where $\Omega_0 : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function.

(H_6) There exists an integrable function $m_2 : J \to [0,\infty)$ such that
$$\|f(t,x,y\| \leq m_2(t)\Omega_1(\|x\| + |y|), \text{ for any } t \in J,\ x,y \in X,$$
where $\Omega_1 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H_7) The function $g : C(J,X) \to X$ is completely continuous and there exists a constant $M_2 > 0$ such that $\|g(x)\| \leq M_2$ for any $x \in X$.

(H_8) The function $\hat{m}(t) = \max\{M_1 m_2(t), m_1(t,t), \int_0^t \frac{\partial m_1(t,s)}{\partial t} ds\}$ satisfies
$$\int_0^T \hat{m}(s)ds < \int_c^\infty \frac{ds}{2\Omega_0(s) + \Omega_1(s)},$$
where $c = M_1[\|x_0\| + M_2]$.

3 Existence of Mild Solutions

The main result is as follows.

THEOREM 3.1. If the assumptions $(H_1) - (H_8)$ are satisfied then the problem (1)-(2) has a mild solution on J.

PROOF. Consider the Banach space $Z = C(J,X)$. We establish the existence of a mild solution of the problem (1)-(2) by applying the Schaefer's fixed point theorem. First we obtain a *priori* bounds for the operator equation
$$x(t) = \lambda \Phi x(t), \quad 0 < \lambda < 1, \tag{3}$$
where $\Phi : Z \to Z$ is defined as
$$(\Phi x)(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)ds. \tag{4}$$

Then from (3) and (4) we have

$$\|x(t)\| \leq M_1[\|x_0\| + M_2] + M_1 \int_0^t m_2(s)\Omega_1(\|x(s)\| + \int_0^s m_2(s,\tau)\Omega_0(\|x(\tau)\|)d\tau)ds.$$

Denoting the right hand side of the above inequality as $v(t)$. Then $\|x(t)\| \leq v(t)$ and $v(0) = c = M_1[\|x_0\| + M_2]$.

$$\begin{aligned} v'(t) &= M_1 m_2(t)\Omega_1(\|x(t)\| + \int_0^t m_1(t,s)\Omega_0(\|x(s)\|)ds) \\ &\leq M_1 m_2(t)\Omega_1(v(t) + \int_0^t m_1(t,s)\Omega_0(v(s))ds), \end{aligned}$$

since v is obviously increasing and let,

$$\begin{aligned} w(t) &= v(t) + \int_0^t m_1(t,s)\Omega_0(v(s))ds. \text{ Then } w(0) = v(0) = c \text{ and } v(t) \leq w(t), \\ w'(t) &= v'(t) + m_1(t,t)\Omega_0(v(t)) + \int_0^t \frac{\partial m_1(t,s)}{\partial t}\Omega_0(v(s))ds \\ &\leq M_1 m_2(t)\Omega_1(w(t)) + m_1(t,t)\Omega_0(w(t)) + \int_0^t \frac{\partial m_1(t,s)}{\partial t}\Omega_0(w(s))ds \\ &\leq \hat{m}(t)\{2\Omega_0(w(t)) + \Omega_1(w(t))\}. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{2\Omega_0(s) + \Omega_1(s)} \leq \int_0^T \hat{m}(s)ds < \int_c^\infty \frac{ds}{2\Omega_0(s) + \Omega_1(s)}, \quad 0 \leq t \leq T. \qquad (5)$$

Inequality (5) implies that there is a constant K such that $v(t) \leq K, t \in J$ and hence we have $\|x\| = \sup\{|x(t)| : t \in J\} \leq K$, where K depends only on T and on the functions $\hat{m}$, Ω_0 and Ω_1.

We shall now prove that the operator $\Phi : Z \to Z$ is a completely continuous operator. Let $B_k = \{x \in Z : \|x\| \leq k\}$ for some $k \geq 1$. We first show that Φ maps B_k into an equicontinuous family.

Let $x \in B_k$ and $t_1, t_2 \in [0,T]$. Then if $0 < t_1 < t_2 < T$,

$$\begin{aligned} &\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\ &\leq \|(R(t_1,0) - R(t_2,0))[x_0 - g(x)]\| \\ &\quad + \|\int_0^{t_1} [R(t_1,s) - R(t_2,s)]f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)ds\| \\ &\quad + \|\int_{t_1}^{t_2} R(t_2,s)f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)ds\| \\ &\leq \|(R(t_1,0) - R(t_2,0))[x_0 - g(x)]\| \\ &\quad + \int_0^{t_1} \|[R(t_1,s) - R(t_2,s)]f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)\|ds \\ &\quad + M_1 \int_{t_1}^{t_2} m_2(s)\Omega_1(k + \int_0^s m_1(s,\tau)\Omega_0(k)d\tau)ds. \end{aligned}$$

The right hand side is independent of $x \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since f is completely continuous and by (H_3), $R(t,s)$ for $t > s$ is continuous in the uniform operator topology . Thus Φ maps B_k into an equicontinuous family of functions.

It is easy to see that ΦB_k is uniformly bounded. Next, we show $\overline{\Phi B_k}$ is compact. Since we have shown ΦB_k is equicontinuous collection, by the Arzela-Ascoli theorem it suffices to show that Φ maps B_k into a precompact set in X.

Let $0 < t \le T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(\Phi_\epsilon x)(t) \quad = \quad R(t,0)[x_0 - g(x)] + \int_0^{t-\epsilon} R(t,s) f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)ds.$$

Since $R(t,s)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every $\epsilon,\ 0 < \epsilon < t$. Moreover, for every $x \in B_k$ we have

$$\begin{aligned} \|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| &\le \int_{t-\epsilon}^t \|R(t,s) f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)\|ds \\ &\le M_1 \int_{t-\epsilon}^t m_2(s)\Omega_1(k + \int_0^s m_1(s,\tau)\Omega_0(k)d\tau)ds. \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_k\}$. Hence, the set $\{(\Phi x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $\Phi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer q such that $\|x_n(t)\| \le q$ for all n and $t \in J$, so $x_n \in B_q$ and $x \in B_q$. By (H_4),

$$f(t, x_n(\sigma_1(t)), \int_0^t k(t,s,x_n(\sigma_2(s)))ds) \to f(t, x(\sigma_1(t)), \int_0^t k(t,s,x(\sigma_2(s)))ds),$$

for each $t \in J$ and since

$$\begin{aligned} &\|f(t, x_n(\sigma_1(t)), \int_0^t k(t,s,x_n(\sigma_2(s)))ds) \\ &\qquad - f(t, x(\sigma_1(t)), \int_0^t k(t,s,x(\sigma_2(s)))ds)\| \le 2m_2(t)\Omega_1(q + \int_0^t m_1(t,s)\Omega_0(q)ds), \end{aligned}$$

we have by dominated convergence theorem

$$\begin{aligned} \|\Phi x_n - \Phi x\| &\le \int_0^t \|R(t,s)[f(s, x_n(\sigma_1(s)), \int_0^s k(s,\tau,x_n(\sigma_2(\tau)))d\tau) \\ &\qquad - f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)]\|ds \\ &\to 0, \ \text{as } n \to \infty. \end{aligned}$$

Thus Φ is continuous. This completes the proof that Φ is completely continuous.

Finally the set $\zeta(\Phi) = \{x \in Z : x = \lambda\Phi x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator Φ has a fixed point in Z. This means that any fixed point of Φ is a mild solution of (1)-(2) on J satisfying $(\Phi x)(t) = x(t)$.

4 Application

As an application of Theorem 3.1 we shall consider the system (1)-(2) with a control parameter such as

$$\begin{aligned} x'(t) &= A(t)x(t)+\int_0^t B(t,s)x(s)ds + Cu(t) \\ &\quad +f(t,x(\sigma_1(t)),\int_0^t k(t,s,x(\sigma_2(s)))ds),\ t\in J \qquad (6) \\ x(0) &+ g(x) = x_0, \qquad (7) \end{aligned}$$

where A, B, f, k, g are as before and C is a bounded linear operator from a Banach space U into X and $u \in L^2(J,U)$. The mild solution of (6)-(7) is given by

$$x(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)[Cu(s) + f(s,x(\sigma_1(s)),\int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)]ds.$$

DEFINITION 4.1. [2] System (6) is said to be controllable with nonlocal condition (7) on the interval J if for every $x_0, x_T \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(\cdot)$ of (6)-(7) satisfies

$$x(0) + g(x) = x_0 \text{ and } x(T) = x_T.$$

To establish the result, we need the following additional conditions:

(H_9) The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^T R(T,s)Cu(s)ds,$$

induces an inverse operator $\tilde{W}^{-1}$ defined on $L^2(J,U)/kerW$ and there exists a positive constant $M_3 > 0$ such that $\|C\tilde{W}^{-1}\| \le M_3$.

(H_{10})The function $\hat{m}(t) = \max\{M_1 m_2(t), m_1(t,t), \int_0^t \frac{\partial m_1(t,s)}{\partial t} ds\}$ satisfies

$$\int_0^T \hat{m}(s)ds < \int_c^\infty \frac{ds}{2\Omega_0(s) + \Omega_1(s)},$$

where c is a constant depending on the system parameters.

THEOREM 4.1. If the hypothesis $(H_1) - (H_7)$ and $(H_9) - (H_{10})$ are satisfied then the system (6) is controllable on J.

PROOF. Using the hypothesis (H_9), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u(t) &= \tilde{W}^{-1}\Big[x_T - R(T,0)[x_0 - g(x)] \\ &\quad - \int_0^T R(T,s)f(s,x(\sigma_1(s)),\int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)ds\Big](t). \end{aligned}$$

We shall show that when using this control, the operator $\Psi : Z \to Z$ defined by

$$\begin{aligned}(\Psi x)(t) &= R(t,0)[x_0 - g(x)] \\ &\quad + \int_0^t R(t,s)[Cu(s) + f(s, x(\sigma_1(s)), \int_0^s k(s,\tau,x(\sigma_2(\tau)))d\tau)]ds,\end{aligned}$$

has a fixed point. This fixed point is, then a solution of (6)-(7). Clearly ,$(\Psi x)(T) = x_T$, which means that the control u steers the system (6)-(7) from the initial state x_0 to x_T in time T, provided we can obtain a fixed point of the nonlinear operator Ψ. The remaining part of the proof is similar to Theorem 3.1, and hence, it is omitted.

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