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Linearized Oscillation Of Even Order Nonlinear Neutral Delay Differential Equations*

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Abstract

In this paper, we proved the even order nonlinear neutral delay differential equation $[x(t) - p(t)g(x(t-\tau))]^{(n)} = q(t)h(x(t-\sigma))$ has the same oscillatory character as its linearized equation $[x(t) - p_0 x(t-\tau)]^{(n)} = q_0 x(t-\sigma)$ under some rather relaxed conditions on $g(u)$ and $h(u)$, where $p_0 = \lim_{t\to\infty} p(t)$, $q_0 = \lim_{t\to\infty} q(t)$.

1 Introduction

Consider the n order nonlinear neutral delay differential equation

$$[x(t) - p(t)g(x(t-\tau))]^{(n)} = q(t)h(x(t-\sigma)), \quad t \geq t_0, \tag{1}$$

where $n \geq 2$ is an even integer,

$$p, q \in C([t_0, \infty), \mathbf{R}), \quad g, h \in C(\mathbf{R}, \mathbf{R}), \quad \tau > 0, \quad \sigma \geq 0. \tag{2}$$

In 1991, Ladas and Qian [8] investigated the equivalence of oscillation for the following special form of Eq.(1)

$$[x(t) - p(t)x(t-\tau)]^{(n)} = q(t)h(x(t-\sigma)), \quad t \geq t_0 \tag{3}$$

and its linearized equation

$$[x(t) - p_0 x(t-\tau)]^{(n)} = q_0 x(t-\sigma), \quad t \geq t_0, \tag{4}$$

and established the following theorem.

THEOREM A ([8]). Assume that

$$0 \leq p(t) \leq p_0 = \lim_{t\to\infty} p(t) < 1, \quad 0 \leq q(t) \leq q_0 = \lim_{t\to\infty} q(t), \quad \text{for large } t; \tag{5}$$

(H_1) $h(u)$ is nondecreasing in the neighborhood of the origin and

$$uh(u) > 0, \quad \text{for } u \neq 0, \quad \text{and} \quad \lim_{u\to 0} \frac{h(u)}{u} = 1;$$

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(H_2) there exists a $\delta > 0$ such that either

$$h(u) \le u, \quad u \in [0, \delta), \qquad \text{or} \qquad h(u) \ge u, \quad u \in (-\delta, 0].$$

Then every bounded solution of Eq.(3) oscillates if and only if every bounded solution of its linearized equation (4) oscillates.

REMARK. The condition (H_2) is missing in the statements of Theorem A in [8], but it is necessary in the proof of Theorem A.

We remark that (H_2) is an additional condition to the linearized condition (H_1), which restricts by u the tendency for functions $h(u)$ to vary in a neighborhood of the origin, and so cause many functions fail to satisfy it. For example, $h(u) = \alpha^{-1}(e^{\alpha u} - 1)$ with $\alpha > 0$. For this reason, [9] tried to remove Condition (H_2), but the example given in [13] shows that (H_2) can not be removed in general. Therefore, it is valuable and necessary to relax the condition (H_2).

In the past 15 years, the linearized oscillation theory for nonlinear neutral delay differential equations has been extensively developed, for example see [1-16]. Linearization is an important method dealt with nonlinear mathematical problems. While the linearized oscillation, roughly speaking, it is find some appropriate hypotheses under which certain nonlinear equations have the same oscillatory character as its associated linearized equation. For Eq.(1), the following linearized oscillation result was obtained by Ladas and Qian in [8] (see also [1, 5]):

THEOREM B ([8]). Assume that

$$\liminf_{t\to\infty} p(t) = p_0 \in (0,1), \qquad p(t) \le P_0 \le 1, \quad \text{for large } t, \tag{6}$$

$$\lim_{t\to\infty} q(t) = q_0 \in (0, \infty), \tag{7}$$

$$0 \le \frac{g(u)}{u} \le \frac{1}{P_0}, \quad \text{for} \quad u \ne 0, \qquad \text{and} \qquad \lim_{u\to 0} \frac{g(u)}{u} = 1, \tag{8}$$

$$uh(u) > 0, \quad \text{for} \quad u \ne 0, \qquad \text{and} \qquad \lim_{u\to 0} \frac{h(u)}{u} = 1. \tag{9}$$

Suppose that every bounded solution of the linearized equation (4) oscillates. Then every bounded solution of Eq.(1) also oscillates.

In this paper, our main purpose is to prove the following theorem which is the converse of Theorem B.

THEOREM C. Assume that there exist $p_0, q_0 \in [0, +\infty)$ such that

$$0 \le p(t) \le p_0 < 1, \quad 0 \le q(t) \le q_0, \quad \text{for large } t, \tag{10}$$

and that the following conditions (H_3) and (H_4) hold:

(H_3) either

$$p(t) > 0 \quad \text{for large } t, \tag{11}$$

or

$$\sigma > 0 \quad \text{and} \quad q(s) \not\equiv 0, \quad s \in [t, t+\sigma], \quad \text{for large } t; \tag{12}$$

(H_4) there exist $r > 0$, $\delta > 0$ and $K > 0$ such that either $g(u)$ and $h(u)$ are nondecreasing in $[0, \delta)$ and

$$0 \le \min\{g(u), h(u)\} \le \max\{g(u), h(u)\} \le u + K|u|^{1+r}, \quad u \in [0, \delta), \tag{13}$$

or $g(u)$ and $h(u)$ are nondecreasing in $(-\delta, 0]$ and

$$0 \ge \max\{g(u), h(u)\} \ge \min\{g(u), h(u)\} \ge u - K|u|^{1+r}, \quad u \in (-\delta, 0]. \tag{14}$$

Suppose also that Eq.(4) has a bounded eventually positive solution. Then Eq.(1) has also a bounded nonoscillatory solution.

By combining Theorems B and C, we have the following two theorems immediately:

THEOREM D. Assume that (5), (9) and (H_4) hold, and that

$$0 \le \frac{g(u)}{u} < \frac{1}{p_0}, \quad \text{for } u \ne 0, \quad \text{and} \quad \lim_{u \to 0} \frac{g(u)}{u} = 1. \tag{15}$$

Then every bounded solution of Eq.(1) oscillates if and only if every bounded solution of its linearized equation (4) oscillates.

THEOREM E. In the assumptions of Theorem A, (H_2) is replaced by the following condition:

(H_5) there exist $r > 0$, $\delta > 0$ and $K > 0$ such that either

$$h(u) \le u + K|u|^{1+r}, \quad u \in [0, \delta),$$

or

$$h(u) \ge u - K|u|^{1+r}, \quad u \in (-\delta, 0].$$

Then every bounded solution of Eq.(3) oscillates if and only if every bounded solution of its linearized equation (4) oscillates.

Obviously, Condition (H_5) is weaker that (H_2). Indeed, (H_5) is a rather relaxed condition. Note that (H_1) implies that $h'(0) = 1$, and so, under (H_1), the function $h(u)$ always satisfy (H_5) whenever $h(u)$ is twice differentiable continuously in some neighborhood of the origin. For example, the function $h(u) = \alpha^{-1}(e^{\alpha u} - 1)$ satisfies (H_5) but not (H_2). Similarly, (H_4) is also a rather relaxed condition. Hence, Theorem D extends and improves Theorem A.

It is a known fact, see [1, 5, 8], that every bounded solution of Eq.(4) oscillates if and only if its characteristic equation

$$F(\lambda) \equiv \lambda^n (1 - p_0 e^{\lambda\tau}) - q_0 e^{\lambda\sigma} = 0 \tag{16}$$

has no nonnegative real roots. Note that

$$F(0) = -q_0 < 0, \quad \text{and} \quad F(\infty) = \lim_{\lambda\to\infty} F(\lambda) = -\infty.$$

So, Eq.(4) has a bounded eventually positive solution implies two possible case: Case (i) there exists a $\lambda^* \in (0,\infty)$ such that

$$F(\lambda^*) > 0; \tag{17}$$

and Case (ii) there exists a $\lambda_0 \in (0,\infty)$ such that

$$F(\lambda_0) = 0 \ \text{ and } \ F(\lambda) \le 0 \ \text{ for } \ \lambda \in [0,\lambda_0) \cup (\lambda_0,\infty). \tag{18}$$

As is customary, if Case (ii) holds, we say Eq.(4) is in a critical state. if Case (i) holds, Eq.(4) is called in a non-critical state. A solution is called oscillatory if it has arbitrary large zeros. Otherwise it is called nonoscillatory.

2 Non-critical Case

In this section, we consider the non-critical case, i.e., case (i) holds.

LEMMA 2.1 ([5]). Every bounded solution of Eq.(4) oscillates if and only if the characteristic equation (16) has no nonnegative real roots.

THEOREM 2.1. Assume that (10), Case (i), (H_3) and the following condition (H_6) hold:
(H_6) there exists a $\delta > 0$ such that either $g(u)$ and $h(u)$ are nondecreasing in $[0,\delta)$ and

$$\min\{g(0),h(0)\} \ge 0, \quad \limsup_{u\to 0^+} \frac{g(u)}{u} \le 1, \quad \limsup_{u\to 0^+} \frac{h(u)}{u} \le 1, \tag{19}$$

or $g(u)$ and $h(u)$ are nondecreasing in $(-\delta,0]$ and

$$\max\{g(0),h(0)\} \le 0, \quad \limsup_{u\to 0^-} \frac{g(u)}{u} \le 1, \quad \limsup_{u\to 0^-} \frac{h(u)}{u} \le 1. \tag{20}$$

Then Eq.(1) has bounded nonoscillatory solution.

PROOF. We only consider the case where (19) in (H_6) holds. The case where (20) in (H_6) holds can be dealt with by a similar fashion. By Case (i), we can choose an $\varepsilon_0 \in (0,1)$ such that $(1+\varepsilon_0)p_0 < 1$ and that

$$\varepsilon_0 \left(\lambda^{*n} p_0 e^{\lambda^* \tau} + q_0 e^{\lambda^* \sigma} \right) < F(\lambda^*). \tag{21}$$

Set

$$F^+_{\varepsilon_0}(\lambda) = \lambda^n \left[1 - (1+\varepsilon_0)p_0 e^{\lambda\tau}\right] - (1+\varepsilon_0) q_0 e^{\lambda\sigma}.$$

Then by (21), we have

$$\begin{aligned} F_{\varepsilon_0}^{+}(\lambda^*) &= \lambda^{*n}\left[1-(1+\varepsilon_0)p_0e^{\lambda^*\tau}\right]-(1+\varepsilon_0)q_0e^{\lambda^*\sigma} \\ &= F(\lambda^*)-\varepsilon_0\left(\lambda^{*n}p_0e^{\lambda^*\tau}+q_0e^{\lambda^*\sigma}\right)>0. \end{aligned}$$

Note that $\lim_{\lambda\to\infty}F_{\varepsilon_0}^{+}(\lambda)=-\infty$, it follows that there exists a $\lambda_1\in(\lambda^*,\infty)$ such that $F_{\varepsilon_0}^{+}(\lambda_1)=0$. By (19) in (H_6), we may choose $\delta_1\in(0,\delta)$ such that

$$0\le\min\{g(0),h(0)\}\le\max\{g(u),h(u)\}\le(1+\varepsilon_0)u,\quad u\in[0,\delta_1). \tag{22}$$

Set $x_1(t)=e^{-\lambda_1 t}$. Then it follows from the fact that $F_{\varepsilon_0}^{+}(\lambda_1)=0$ that

$$[x_1(t)-(1+\varepsilon_0)p_0x_1(t-\tau)]^{(n)}=(1+\varepsilon_0)q_0x_1(t-\sigma),\qquad t\ge t_0, \tag{23}$$

which yields that

$$x_1(t)=(1+\varepsilon_0)p_0x_1(t-\tau)+\frac{(1+\varepsilon_0)q_0}{(n-1)!}\int_t^\infty(s-t)^{n-1}x_1(s-\sigma)ds,\qquad t\ge t_0. \tag{24}$$

Choose $T>t_0$ such that $x_1(t-\tau-\sigma)<\delta_1$ for $t\ge T$, and both (10) and (H_3) hold for $t\ge T$. Then from (10), (22) and (24), we obtain

$$x_1(t)\ge p(t)g(x_1(t-\tau))+\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)h(x_1(s-\sigma))ds,\qquad t\ge T. \tag{25}$$

Hence it follows in a manner similar to that in the proof of [3, Lemma 5.1.5] that the corresponding integral equation

$$x(t)=p(t)g(x(t-\tau))+\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)h(x(s-\sigma))ds,\qquad t\ge T \tag{26}$$

has a solution $x_2(t)$ with $0<x_2(t)\le x_1(t)$ for $t\ge T$. Clearly, $x_2(t)$ is also bounded eventually positive solution of Eq.(1). The proof is complete.

3 Critical case

In this section, we consider the critical case, i.e., case (ii) holds.

THEOREM 3.1. Assume that (10), Case (ii), (H_3) and (H_4) hold. Then Eq.(1) has bounded nonoscillatory solution.

PROOF. We only consider the case where (13) in (H_4) holds. The case where (14) in (H_4) holds is similar and so omit it. It follows from Case (ii) that

$$F(\lambda_0)=0\ \text{ and }\ F'(\lambda_0)=0,$$

which yields that

$$\lambda_0^n=\lambda_0^np_0e^{\lambda_0\tau}+q_0e^{\lambda_0\sigma} \tag{27}$$

and

$$n\lambda_0^{n-1} = n\lambda_0^{n-1}p_0e^{\lambda_0\tau} + \lambda_0^n p_0\tau e^{\lambda_0\tau} + q_0\sigma e^{\lambda_0\sigma}. \tag{28}$$

From (27) and (28), it is easy to see that $\lambda_0 > 0$ and that

$$(n-\lambda_0\sigma)\left(1-p_0e^{\lambda_0\tau}\right) = \lambda_0\tau p_0 e^{\lambda_0\tau}. \tag{29}$$

Set $x_0(t) = \sqrt{t}e^{-\lambda_0 t}$. Then from (27), (28) and (29), we have

$$\begin{aligned}
& [x_0(t)-p_0x_0(t-\tau)]^{(n)} - q_0x_0(t-\sigma) \\
= \ & \left(\sqrt{t}e^{-\lambda_0 t}\right)^{(n)} - p_0e^{\lambda_0\tau}\left(\sqrt{t-\tau}e^{-\lambda_0 t}\right)^{(n)} - q_0e^{\lambda_0\sigma}\sqrt{t-\sigma}e^{-\lambda_0 t} \\
= \ & e^{-\lambda_0 t}\Bigg(-q_0e^{\lambda_0\sigma}\sqrt{t-\sigma} + \Bigg[\lambda_0^n\sqrt{t} - \frac{n\lambda_0^{n-1}}{2\sqrt{t}} - \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}} \\
& \quad -\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg] \\
& -p_0e^{\lambda_0\tau}\Bigg[\lambda_0^n\sqrt{t-\tau} - \frac{n\lambda_0^{n-1}}{2\sqrt{t-\tau}} - \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}} \\
& \quad -\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\Bigg]\Bigg) \\
= \ & e^{-\lambda_0 t}\Bigg(\Bigg[\frac{q_0\sigma e^{\lambda_0\sigma}}{\sqrt{t}+\sqrt{t-\sigma}} - \frac{n\lambda_0^{n-1}}{2\sqrt{t}} - \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}} \\
& \quad -\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg] \\
& +p_0e^{\lambda_0\tau}\Bigg[\frac{\lambda_0^n\tau}{\sqrt{t}+\sqrt{t-\tau}} + \frac{n\lambda_0^{n-1}}{2\sqrt{t-\tau}} + \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}} \\
& \quad +\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\Bigg]\Bigg) \\
= \ & e^{-\lambda_0 t}\Bigg(\Bigg[\frac{q_0\sigma^2 e^{\lambda_0\sigma}}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\sigma})^2} - \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}} \\
& \quad -\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg] \\
& +p_0e^{\lambda_0\tau}\Bigg[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2} + \frac{n\lambda_0^{n-1}\tau}{2\sqrt{t}\sqrt{t-\tau}(\sqrt{t}+\sqrt{t-\tau})} + \frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}} \\
& \quad +\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\Bigg]\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\geq\ & e^{-\lambda_0 t}\Bigg(\frac{1}{8\sqrt{t^3}}\Big[q_0\sigma^2 e^{\lambda_0\sigma} + p_0 e^{\lambda_0\tau}(\lambda_0^n\tau^2 + 2n\lambda_0^{n-1}\tau) - n(n-1)\lambda_0^{n-2}\left(1-p_0e^{\lambda_0\tau}\right)\Big] \\
& -\left(1-p_0e^{\lambda_0\tau}\right)\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg) \\
=\ & e^{-\lambda_0 t}\Bigg(\frac{1}{8\sqrt{t^3}}\Big[\lambda_0^n\left(1-p_0e^{\lambda_0\tau}\right)\sigma^2 + p_0 e^{\lambda_0\tau}(\lambda_0^n\tau^2 + 2n\lambda_0^{n-1}\tau) \\
& \quad -n(n-1)\lambda_0^{n-2}\left(1-p_0e^{\lambda_0\tau}\right)\Big] \\
& \quad -\left(1-p_0e^{\lambda_0\tau}\right)\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg) \\
=\ & e^{-\lambda_0 t}\Bigg(\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\Big[\left(1-p_0e^{\lambda_0\tau}\right)\left((\lambda_0\sigma)^2 - n(n-1)\right) + p_0e^{\lambda_0\tau}\left((\lambda_0\tau)^2 + 2n(\lambda_0\tau)\right)\Big] \\
& \quad -\left(1-p_0e^{\lambda_0\tau}\right)\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg) \\
=\ & e^{-\lambda_0 t}\Bigg(\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\Bigg[\left(1-p_0e^{\lambda_0\tau}\right)\left(\left(n-\frac{\lambda_0\tau p_0e^{\lambda_0\tau}}{1-p_0e^{\lambda_0\tau}}\right)^2 - n(n-1)\right) \\
& \quad +p_0e^{\lambda_0\tau}\left((\lambda_0\tau)^2 + 2n(\lambda_0\tau)\right)\Bigg] \\
& \quad -\left(1-p_0e^{\lambda_0\tau}\right)\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg) \\
=\ & e^{-\lambda_0 t}\Bigg(\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\Bigg[\frac{(\lambda_0\tau)^2\left(p_0e^{\lambda_0\tau}\right)^2}{1-p_0e^{\lambda_0\tau}} + n\left(1-p_0e^{\lambda_0\tau}\right) + (\lambda_0\tau)^2 p_0e^{\lambda_0\tau}\Bigg] \\
& \quad -\left(1-p_0e^{\lambda_0\tau}\right)\sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg) \\
\geq\ & \left(1-p_0e^{\lambda_0\tau}\right)e^{-\lambda_0 t}\Bigg[\frac{n\lambda_0^{n-2}}{8\sqrt{t^3}} - \sum_{k=3}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg].
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
[x_0(t-\sigma)]^{1+r} &= \left(\sqrt{t-\sigma}\,e^{-\lambda_0(t-\sigma)}\right)^{1+r} \\
&\leq t^{(1+r)/2}e^{-(1+r)\lambda_0(t-\sigma)} \\
&= t^{(1+r)/2}e^{-\lambda_0[rt-(1+r)\sigma]}e^{-\lambda_0 t},
\end{aligned}
$$

and for large t,

$$\begin{aligned}
& \left\{[x_0(t-\tau)]^{1+r}\right\}^{(n)} \\
= \;& e^{(1+r)\lambda_0\tau}\left[(t-\tau)^{(1+r)/2}e^{-(1+r)\lambda_0 t}\right]^{(n)} \\
= \;& e^{-(1+r)\lambda_0(t-\tau)}\left[(1+r)^n\lambda_0^n(t-\tau)^{(1+r)/2}-\frac{n(1+r)^n\lambda_0^{n-1}}{2(t-\tau)^{(1-r)/2}}\right. \\
& \left.-\sum_{k=2}^{n}\frac{n(n-1)\cdots(n-k+1)\cdot(1-r)\cdot(3-r)\cdots(2k-3-r)(1+r)^{n-k+1}\lambda_0^{n-k}}{2^k k!(t-\tau)^{(2k-1-r)/2}}\right] \\
\le \;& 2(1+r)^n\lambda_0^n t^{(1+r)/2}e^{-(1+r)\lambda_0(t-\tau)}.
\end{aligned}$$

Thus, there exists a large $T>t_0$ such that $x_0(t-\tau-\sigma)<\delta$ for $t\ge T$, and both (10) and (H_3) hold for $t\ge T$ and

$$[x_0(t)-p_0x_0(t-\tau)]^{(n)}-q_0x_0(t-\sigma)\ge p_0K\left\{[x_0(t-\tau)]^{1+r}\right\}^{(n)}+q_0K|x_0(t-\sigma)|^{1+r}, \tag{30}$$

for $t\ge T$. It follows from (10), (30) and (H_4) that

$$\begin{aligned}
x_0(t) \;\ge\;& p_0\left[x_0(t-\tau)+K|x_0(t-\tau)|^{1+r}\right] \\
& +\frac{q_0}{(n-1)!}\int_t^\infty(s-t)^{n-1}\left[x_0(s-\sigma)+K|x_0(s-\sigma)|^{1+r}\right]ds \\
\ge\;& p(t)g(x_0(t-\tau))+\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)h(x_0(s-\sigma))ds, \qquad t\ge T.
\end{aligned}$$

This shows that the inequality

$$x(t)\ge p(t)g(x(t-\tau))+\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)h(x(s-\sigma))ds, \qquad t\ge T. \tag{31}$$

has a positive solution $x_0(t)$. Similar to the proof of Theorem 2.1, it is easy to show that Eq.(1) has a bounded eventually positive solution. The proof is complete.

4 Some Remarks

From Theorems 2.1 and 3.1, we have immediately Theorem C.

EXAMPLE 4.1[13]. Consider the following nonlinear equation

$$x''(t)=\frac{4}{e^2}[x(t-1)+f(x(t-1))], \qquad t\ge 0 \tag{32}$$

and its corresponding linear equation

$$x''(t)=\frac{4}{e^2}x(t-1), \qquad t\ge 0 \tag{33}$$

where $n = 2$, $q(t) \equiv q_0 = \frac{4}{e^2}$, $\sigma = 1$, and

$$f(u) = \begin{cases} 0, & u = 0, \\ \frac{u}{1-\ln|u|}, & 0 < |u| \leq 1, \\ u, & |u| > 1. \end{cases} \tag{34}$$

It is easy to see that Eq.(33) is in a critical state, and so Eq.(33) has a bounded eventually positive solution $x_0(t) = e^{-2t}$, but in view of the proof in [13], every bounded solution of Eq.(32) oscillates. Consequently, (32) and (33) have different oscillatory behavior.

If we try to examine the cause, we see that (32) does not satisfies (H_5). In fact, for $r > 0$,

$$\lim_{u\to 0} \frac{|f(u)|}{|u|^{1+r}} = \lim_{u\to 0} \frac{1}{|u|^r(1-\ln|u|)} = \infty.$$

The above example shows that (H_5) is an essential condition that guarantees that (3) and (4) have the same oscillatory behavior in the critical case.

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