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Dynamical Behavior Of A Third-Order Rational Difference Equation*

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Abstract

We investigate the dynamical behavior of the following third-order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} + x_n + x_{n-1} + x_{n-2} + a}{x_n x_{n-1} + x_n x_{n-2} + x_{n-1} x_{n-2} + 1 + a}, \quad n = 0, 1, 2, \ ...,$$

where $a \in [0, \infty)$ and the initial values $x_{-2},\ x_{-1},\ x_0 \in (0, \infty)$. We find that the successive lengths of positive and negative semicycles of nontrivial solutions of the above equation occur periodically. We also show that the positive equilibrium of the equation is globally asymptotically stable.

1 Introduction

Ladas [3] proposed to study the rational difference equation

$$x_{n+1} = \frac{x_n + x_{n-1} x_{n-2} + a}{x_n x_{n-1} + x_{n-2} + a}, \quad n = 0, 1, 2, ... \tag{1}$$

From then on, rational difference equations with the unique positive equilibrium $\overline{x} = 1$ have received considerable attention, one can refer to [3-5, 7, 14-16, 18, 20] and the references cited therein.

Recently, Li [4] investigated the global behavior of the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ..., \tag{2}$$

where $a \in [0, \infty)$ and initial value $x_{-3},\ x_{-2},\ x_{-1},\ x_0 \in (0, \infty)$.

From Li [4] , we know that the successive lengths of positive and negative semicycles of nontrivial solutions of Eq. (2) is periodic. By means of this, the positive equilibrium of Eq. (2) is shown to be globally asymptotically stable.

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In this note, we employ the method in Li [4] to consider the following third-order rational difference equations

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} + x_n + x_{n-1} + x_{n-2} + a}{x_n x_{n-1} + x_n x_{n-2} + x_{n-1} x_{n-2} + 1 + a}, \quad n = 0, 1, 2, \ ..., \tag{3}$$

where $a \in [a, \infty)$, and $x_{-2},\ x_{-1},\ x_0 \in (0, \infty)$.

By analyzing the lengths of positive and negative semicycles of nontrivial solutions of Eq. (3), we find that the lengths of positive and negative semicycles of nontrivial solutions of Eq. (3) occur periodically and can be expressed in the "form": $..., 2^-,\ 2^+,\ 2^-,\ 2^+,\ 2^-,\ 2^+,\ 2^-, 2^+,\ ...,$ or $1^-,\ 1^+,\ 1^-,\ 1^+,\ 1^-,\ 1^+,\ 1^-, 1^+,\ ....$

What we obtain is different from the ones in the literature [4] and enables us to show that the positive equilibrium of Eq. (3) is globally asymptotically stable.

To our best knowledge, Eq. (3) has not been investigated so far. Therefore, to study the global behavior of its positive solutions is meaningful and interesting.

It is easy to see that the positive equilibrium $\overline{x}$ of Eq. (3) satisfies

$$\overline{x} = \frac{\overline{x}^3 + 3\overline{x} + a}{3\overline{x}^2 + 1 + a}$$

from which one can see that Eq. (3) has a unique positive equilibrium $\overline{x} = 1$.

DEFINITION 1.1. A positive semicycle of a solution $\{x_n\}_{n=-2}^{\infty}$ of equation Eq. (3) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -2$ and $m \leq \infty$ such that either $l = -2$ or $l > -2$ and $x_{l-1} < \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$. A negative semicycle of a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (3) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than $\overline{x}$, with $l \geq -2$ and $m \leq \infty$ such that either $l = -2$ or $l > -2$ and$x_{l-1} \geq \overline{x}$, and either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

The length of a semicycle is the number of the total terms contained in it.

DEFINITION 1.2. A solution $\{x_n\}_{-2}^{\infty}$ of Eq. (3) is said to be eventually trivial if x_n is eventually equal to $\overline{x} = 1$; otherwise, the solution is said to be nontrivial.

DEFINITION 1.3. A sequence $\{x_n\}_{n=0}^{\infty}$ is called strictly oscillatory if for every $n_0 \geq 0$, there exist $n_{1,}\ n_2 \geq n_0$, such that $x_{n_1} x_{n_2} < 0$. A sequence is called strictly oscillatory about $\overline{x}$ if the sequence $\{x_n - \overline{x}\}$ is strictly oscillatory.

2 Two lemmas

In this section, we establish two lemmas which will be useful in the proof of our main results.

LEMMA 2.1. A positive solution of Eq. (3) is eventually equal to 1 if and only if

$$(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0. \tag{4}$$

PROOF. Assume that (4) holds. Then, according to Eq. (3) we can obtain the following conclusions:

(1) if $x_{-2} = 1$, then $x_n = 1$ for $n \geq 1$;
(2) if $x_{-1} = 1$, then $x_n = 1$ for $n \geq 1$;
(3) if $x_0 = 1$, then $x_n = 1$ for $n \geq 0$.
Conversely, assume that

$$(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0.$$

Then, one can show that $x_n \neq 1$ for any $n \geq 1$. In fact, assume the contrary that for some $N \geq 1$, such that

$$x_N = 1 \text{ and that } x_n \neq 1 \text{ for } -2 \leq n \leq N-1. \tag{5}$$

It is easy to see that

$$1 = x_N = \frac{x_{N-1}x_{N-2}X_{N-3} + X_{N-1} + X_{N-2} + X_{N-3} + a}{x_{N-1}X_{N-2} + X_{N-1}X_{N-3} + X_{N-2}X_{N-3} + 1 + a},$$

which implies that $(x_{N-3} - 1)(x_{N-2} - 1)(x_{N-1} - 1) = 0$. This is contrary to (5).

REMARK 2.1. If the initial conditions of Eq. (3) do not satisfy (4), then, for any solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (3), $x_n \neq 1$ for $n \geq -2$. Hence, the solution is a nontrivial one.

LEMMA 2.2. Let $\{x_n\}_{n=-2}^{\infty}$ be nontrivial positive solution of Eq. (3). Then the following four conclusions are true for $n \geq 0$:
(a) $(x_{n+1} - 1)(x_n - 1)(x_{n-1} - 1)(x_{n-2} - 1) > 0$;
(b) $(x_{n+1} - x_n)(x_n - 1) < 0$;
(c) $(x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0$;
(d) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$.

PROOF. It follows from Eq. (3) that

$$x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-1} - 1)(x_{n-2} - 1)}{x_n x_{n-1} + x_n x_{n-2} + x_{n-1}x_{n-2} + 1 + a}, \quad n = 0, 1, 2, ...,$$

and

$$x_{n+1} - x_n = \frac{(1 - x_n)[x_{n-1}(1 + x_n) + x_{n-2}(1 + x_n) + a]}{x_n x_{n-1} + x_n x_{n-2} + x_{n-1}x_{n-2} + 1 + a}, \quad n = 0, 1, 2, ...,$$

from which inequalities (a) and (b) follow. The proofs for inequalities (c) and (d) are similar to the one for inequality (b).

3 Main results

First we analyze the trajectory structure of the semicycles of nontrivial solutions of Eq. (3). Here, we confine us to consider the situation of the strictly oscillatory solutions about $\overline{x} = 1$ of Eq. (3).

THEOREM 3.1. Let $\{x_n\}_{n=-2}^{\infty}$ be a strictly oscillatory solutions of Eq. (3). Then the "rule of the trajectory structure" of nontrivial solutions of Eq. (3) is: ..., 2^-, 2^+, 2^-, 2^+, 2^-, 2^+, 2^-, 2^+, ... or,..., 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, ...

PROOF. By Lemma 2.2(a) and the character of the strictly oscillatory one can see the lengths of a positive or a negative semicycle is at most 2. So, for some integer $p \geq 0$, one of the following two cases must occur:

Case 1. $x_{p-2} < 1$, $x_{p-1} > 1$ and $x_p > 1$.

Case 2. $x_{p-2} < 1$, $x_{p-1} > 1$ and $x_p < 1$.

If Case 1 occurs, it follows from Lemmas 2.2 (a) that

$x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} > 1$, $x_{p+5} < 1, x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1, x_{p+9} < 1$, $x_{p+10} < 1$,

$x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1, x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$,

$x_{p+20} > 1$, $x_{p+21} < 1$, $x_{p+22} < 1$, $x_{p+23} > 1$, $x_{p+24} > 1, x_{p+25} < 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} > 1$, $x_{p+29} < 1$,

$x_{p+30} < 1, x_{p+31} > 1, x_{p+32} > 1, x_{p+33} < 1, x_{p+34} < 1, x_{p+35} > 1, x_{p+36} > 1, x_{p+37} < 1, x_{p+38} < 1, x_{p+39} > 1, x_{p+40} > 1$,

$x_{p+41} < 1, x_{p+42} < 1, \ldots.$

If Case 2 occurs, then Lemma 2.2 (a) implies that

$x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} < 1$, $x_{p+5} > 1$, $x_{p+6} < 1, x_{p+7} > 1$, $x_{p+8} < 1$, $x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} > 1$, $x_{p+12} < 1$,

$x_{p+13} > 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} > 1$, $x_{p+22} < 1$, $x_{p+23} > 1$,

$x_{p+24} < 1$, $x_{p+25} > 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} > 1, x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} < 1$, $x_{p+33} > 1$, $x_{p+34} < 1$,

$x_{p+35} > 1$, $x_{p+36} < 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} > 1$, $x_{p+42} < 1$, $x_{p+43} > 1$, $x_{p+44} < 1, \ldots.$

We may now see that the lengths of the positive and negative semicycles is either of the form ..., 2^-, 2^+, 2^-, 2^+, 2^-, 2^+, $2^-, 2^+$, ... or of the form ..., 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, $1^-, 1^+$, ..., where 2^- stands for the length of a negative semicycle is 2, etc.

REMARK 3.1. It is easy to see that the cases in the proof of Theorem 3.1 are caused by the perturbation of the initial values around the equilibrium $\overline{x} = 1$. So, Theorem 3.1 indicates that the perturbation of the initial values may lead to the variation of the trajectory structure rule for the solutions of Eq. (3).

THEOREM 3.2. Assume that $a \in [0, \infty)$. Then the positive equilibrium of Eq. (3) is globally asymptotically stable.

PROOF. We must prove that the positive equilibrium of Eq. (3) is both locally asymptotically stable and globally attractive. The linearized equation of Eq. (3) about the positive equilibrium $\overline{x} = 1$ is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2}. \qquad n = 0, 1, 2, \ldots$$

By virtue of [2, Remark 1.3.1], $\overline{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (3) converges to 1 as $n \to \infty$. Namely, we want to prove

$$\lim_{n \to \infty} x_n = 1 \tag{6}$$

If the initial values of the solutions satisfy (4), then Lemma 2.1 says the solution is eventually equal to 1, and, of course, (6) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (4). Then by Remark 2.1, we know, for any solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (3), $x_n \neq 1$ for $n \geq -2$.

If the solution is nonoscillatory about the positive equilibrium point $\overline{x}$ of Eq. (3), then we know from Lemma 2.2(b), the solution is monotonic and bounded. So, the limit $\lim_{n\to\infty} x_n = L$ exists and is finite. Taking the limit on both sides of Eq. (3), we obtain

$$L = \frac{L^3 + 3L + a}{3L^2 + 1 + a}.$$

Solving this equation gives rise to $L = 1$, which shows (6) is true.

Thus, it suffices to prove that Eq. (6) holds for the solution which is strictly oscillatory.

Consider now $\{x_n\}_{n=-2}^{\infty}$ strictly oscillatory about the positive equilibrium $\overline{x}$ of Eq. (3). By virtue of Theorem 3.1, one understands that the lengths of positive and negative semicycles which occur successively is ..., $2^-, 2^+, 2^-, 2^+, 2^-, 2^+, 2^-, 2^+$, ... or ..., 1^-, $1^+, 1^-, 1^+, 1^-, 1^+, 1^-, 1^+$, ...

First, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is..., $2^-, 2^+, 2^-, 2^+, 2^-, 2^+, 2^-, 2^+$, ...

For the sake of convenience, we denote by $\{x_p, x_{p+1}\}^-$ and $\{x_{p+2}, x_{p+3}\}^+$ the terms of a negative semicycle and a positive semicycle of length two respectively. So, the rule for the negative and positive semicycles to occur successively can be periodically expressed as follows:

$$\{\ x_{p+4n}, x_{p+4n+1}\}^-, \quad \{x_{p+4n+2}, x_{p+4n+3}\}^+, \qquad n = 0, 1, 2, ...$$

The following results (1) and (2) can be easily observed:

(1) $x_{p+4n} < x_{p+4n+1} < x_{p+4n+4}$;

(2) $x_{p+4n+6} < x_{p+4n+3} < x_{p+4n+2}$.

In fact, the above inequalities (1) and (2) follow directly from Lemma 2.2(b) and Lemma 2.2(d), respectively.

We can see from inequality (1) and (2) that $\{x_{p+4n}\}_{n=0}^{\infty}$ is increasing with upper bound 1 and $\{x_{p+4n+2}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So the limits

$$\lim_{n\to\infty} x_{p+4n} = L \quad \text{and} \quad \lim_{n\to\infty} x_{p+4n+2} = M$$

exist and are finite. Furthermore, in light of (1) and (2) we can get

$$\lim_{n\to\infty} x_{p+4n+1} = L$$

and

$$\lim_{n\to\infty} x_{p+4n+3} = M$$

Noting that

$$x_{p+4n+3} = \frac{x_{p+4n+2}x_{4n+1}x_{4n} + x_{p+4n+2} + x_{p+4n+1} + x_{p+4n} + a}{x_{p+4n+2}x_{p+4n+1} + x_{p+4n+2}x_{p+4n} + x_{p+4n+1}x_{p+4n} + 1 + a}.$$

and taking the limit on both sides of this equality gives rise to

$$M = \frac{ML^2 + M + 2L + a}{2ML + L^2 + 1 + a}.$$

Solving this equation we get $M = 1$.

So,

$$\lim_{n\to\infty} x_{p+4n+2} = \lim_{n\to\infty} x_{p+4n+3} = 1.$$

Again taking the limit on both sides of the following equation

$$x_{p+4n+4} = \frac{x_{p+4n+3}x_{4n+2}x_{4n+1} + x_{p+4n+3} + x_{p+4n+2} + x_{p+4n+1} + a}{x_{p+4n+3}x_{4n+2} + x_{p+4n+3}x_{4n+1} + x_{p+4n+2}x_{4n+1} + 1 + a},$$

we get

$$L = \frac{L + 1 + 1 + L + a}{1 + L + L + 1 + a} = 1.$$

So,

$$\lim_{n\to\infty} x_{p+4n} = \lim_{n\to\infty} x_{p+4n+1} = 1$$

Thus,

$$\lim_{n\to\infty} x_{p+4n+k} = 1, \quad k = 0, 1, 2, 3.$$

Namely,

$$\lim_{n\to\infty} x_n = 1.$$

Second, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is..., 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, ...

Similar to the first case, the rule for the positive and negative semicycles to occur successively can be periodically expressed as follows:

$$\{x_{p+2n}\}^-, \quad \{x_{p+2n+1}\}^+, \qquad n = 0, 1, 2, ...$$

The following results can be easily obtained from Lemma 2.2(c):

(1) $x_{p+2n} < x_{p+2n+2}$;

(2) $x_{p+2n+3} < x_{p+2n+1}$.

From these one can see $\{x_{p+2n}\}_{n=0}^{\infty}$ is increasing with upper 1 and $\{x_{p+2n+1}\}_{n=0}^{\infty}$ is decreasing with lower 1, therefore,

$$\lim_{n\to\infty} x_{p+2n} = L,$$

and

$$\lim_{n\to\infty} x_{p+2n+1} = M$$

exist and are finite.

Taking the limit on both sides of the following equality

$$x_{p+2n+3} = \frac{x_{p+2n+2}x_{p+2n+1}x_{p+2n} + x_{p+2n+2} + x_{p+2n+1} + x_{p+2n} + a}{x_{p+2n+2}x_{p+2n+1} + x_{p+2n+2}x_{p+2n} + x_{p+2n+1}x_{p+4n} + 1 + a}$$

we get

$$M = \frac{ML^2 + M + 2L + a}{LM + L^2 + ML + 1 + a},$$

which yields $M = 1$. So,

$$\lim_{n\to\infty} x_{p+2n+1} = 1.$$

Again, taking the limit on both sides of the following equality

$$x_{p+2n+4} = \frac{x_{p+2n+3}x_{p+2n+2}x_{p+2n+1} + x_{p+2n+3} + x_{p+2n+2} + x_{p+2n+1} + a}{x_{p+2n+3}x_{p+2n+2} + x_{p+2n+3}x_{p+4n+1} + x_{p+2n+2}x_{p+4n+1} + 1 + a},$$

we get

$$L = \frac{L + 1 + L + 1 + a}{L + 1 + L + 1 + a} = 1.$$

So,

$$\lim_{n\to\infty} x_{2n} = 1.$$

We have shown

$$\lim_{n\to\infty} x_n = 1,$$

which completes our proof.

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