Applied Mathematics E-Notes, 6(2006), 289-293 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Convergence Theorem For Mann Fixed Point Iteration Procedure*

Arif Rafiq†

Received 11 November 2005

Abstract

We establish a general theorem to approximate fixed points of z-operators on a normed space through the Mann iteration process with errors in the sense of Liu [8]. Our result generalizes and improves upon, among others, the corresponding result of Rhoades [12].

Throughout this note, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping.

The Mann iteration process is defined by the sequence $\{x_n\}$ (see [9]):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n, \ n \in \mathbb{N} \end{cases} \tag{1}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

Liu [8] introduced the concept of Mann iteration process with errors by the sequence $\{x_n\}$ defined as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n + u_n, \ n \in \mathbb{N} \end{cases} . \tag{2}$$

where $\{b_n\}$ is a sequence in $[0, 1]$ and $\{u_n\}$ satisfy $\sum_{n=1}^{\infty} \|u_n\| < \infty$. This surely contains (1).

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X, \tag{3}$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [5] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \ \text{ for all } x, y \in X. \tag{4}$$

*Mathematics Subject Classifications: 47H10, 47H17, 54H25.

†COMSATS Institute of Information Technology, Islamabad, Pakistan

A similar definition is due to Chatterjea [2]: there exists a $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{5}$$

Combining these three definitions, Zamfirescu [17] proved the following important result.

THEOREM 1. Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for each pair $x, y \in X$, at least one of the following conditions holds:

(z_1) $d(Tx, Ty) \leq ad(x, y)$,
(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,
(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

One of the most general contraction condition, for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [3] in 1974: there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}, \tag{QC}$$

for all $x, y \in X$.

REMARK 1. 1. A mapping satisfying (QC) is commonly called quasi contraction. It is obvious that each of the conditions (1)-(3) and (z_1)-(z_3) implies (QC). 2. An operator T satisfying the contractive conditions (z_1)-(z_3) in the above theorem is called z-operator.

The following lemma is proved in [11].

LEMMA 2. Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

In this paper, a convergence theorem of Rhoades [12] regarding the approximation of fixed points of z-operators in uniformly convex Banach spaces using the Mann iteration process, is extended to arbitrary normed spaces using the Mann iteration process with errors in the sense of Liu [8]. The conditions on the parameters $\{b_n\}$ that define the Mann iteration are also weakened.

THEOREM 3. Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an z-operator. Let $\{x_n\}$ be defined by the iterative process (2). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\|u_n\| = 0(b_n)$, then $\{x_n\}$ converges strongly to a fixed point of T.

PROOF. By Theorem 1, we know that T has a unique fixed point in C, say w. Consider $x, y \in C$. Following the approach of Berinde [1], since T is a z-operator, at least one of the conditions $(z_1), (z_2)$ and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned}\|Tx - Ty\| &\leq b\left[\|x - Tx\| + \|y - Ty\|\right] \\ &\leq b\left[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right],\end{aligned}$$

implies

$$(1 - b)\|Tx - Ty\| \leq b\|x - y\| + 2b\|x - Tx\|,$$

which yields (using the fact that $0 \leq b < 1$)

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|. \tag{6}$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\|. \tag{7}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{8}$$

Then we have $0 \leq \delta < 1$ and, in view of (z_1), (6)-(8) it results that the inequality

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \tag{9}$$

holds for all $x, y \in C$.

Using (2), we have

$$\begin{aligned}\|x_{n+1} - w\| &= \|(1 - b_n)x_n + b_n Ty_n + u_n - w\| \\ &= \|(1 - b_n)(x_n - w) + b_n(Ty_n - w) + u_n\| \\ &\leq (1 - b_n)\|x_n - w\| + b_n\|Ty_n - w\| + \|u_n\|. \end{aligned} \tag{10}$$

Now for $x = w$ and $y = y_n$, (9) gives

$$\|Ty_n - w\| \leq \delta\|y_n - w\|, \tag{11}$$

and hence, by (10)-(11) we obtain

$$\|x_{n+1} - w\| \leq [1 - (1 - \delta)b_n]\|x_n - w\| + \|u_n\|,\ n = 0, 1, 2, \ldots.$$

With the help of Lemma 1 and using the fact that $0 \leq \delta < 1$, $0 \leq b_n \leq 1$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\|u_n\| = 0(b_n)$, it results that $\lim_{n\to\infty}\|x_{n+1} - w\| = 0$. Consequently $x_n \to w \in F$ and this completes the proof.

COROLLARY 4. Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (9). Let $\{x_n\}$ be defined by the iterative process (1). If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.

REMARK 2.

1. The contractive condition (3) makes T a continuous function on X while this is not the case with the contractive conditions (4)-(5) and (9).

2. The Chatterjea's and the Kannan's contractive conditions (5) and (4) are both included in the class of Zamfirescu operators and so their convergence theorems for the simple Mann iteration process are obtained in Corollary 1.

3. Theorem 4 of Rhoades [12] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 1.

4. In Corollary 1, Theorem 8 of Rhoades [13] is generalized to the setting of normed spaces.

EXAMPLE 1. An example of the mapping satisfied (9) is as follows [1]: Let $X = \mathbb{R}$ and $T : X \to X$, $T(x) = 0$, if $x \in (-\infty, 2]$ and $T(x) = -\frac{1}{2}$, if $x > 2$.

References

[1] V. Berinde, On the convergence of the Ishikawa iteration in the class of quasi contractive operators, Acta Math Univ. Comenianae, 7391)(2004), 119-126.

[2] S. K.Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci., 25(1972), 727–730.

[3] L. B. Ciric, A generalization of Banach principle, Proc. Amer. Math. Soc., 45(1974), 727–730.

[4] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc., 44(1974), 147–150.

[5] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 10(1968), 71–76.

[6] R. Kannan, Some results on fixed points III, Fund. Math.,70(1971), 169–177.

[7] R. Kannan, Construction of fixed points of class of nonlinear mappings, J. Math. Anal. Appl., 41(1973), 430–438.

[8] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl., 194(1)(1995), 114–125.

[9] W. R. Mann, Mean value methods in iterations, Proc. Amer. Math. Soc., 4 (1953), 506–510.

[10] M. O. Osilike, Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings, Indian.J. Pure and Appl. Math., 30(12)1999, 1229–1234.

[11] M. O. Osilike, Stability results for fixed point iteration procedures, J. Nigerian Math.Soc., 14/15(1995/1996), 17–29.

[12] B. E. Rhoades, Fixed point iteration using infinite matrices, Trans. Amer. Math.Soc., 196(1974), 161–176.

[13] B. E. Rhoades, Comments on two fixed point iteration method, J. Math. Anal. Appl., 56(2)(1976), 741–750.

[14] W. Takahashi, Iterative methods for approximation of fixed points and thier applications, J. Oper. Res. Soc. Jpn., 43(1)(2000), 87–108.

[15] W. Takahashi and T. Tamura, Convergence theorems for a pair of nonexpansive mappings, J. Convex Analysis, 5(1)(1995), 45–58.

[16] Y. Xu, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl., 224(1998), 91–101.

[17] T. Zamfirescu, Fixed point theorems in metric spaces, Arch. Math. (Basel), 23(1972), 292–298.