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Existence Criteria For Singular Boundary Value Problems With p-Laplacian Operators*

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Abstract

The purpose of this paper is to prove some existence theorems for certain classes of singular nonlinear two point boundary value problems with p-Laplacian operators. In our study, we shall use nonlinear alternative theorem of Leray-Schauder type to prove a priori bound theorem, and from the theorem we get two existence results of differential equations without growth restrictions.

In recent years, singular two-point boundary value problems for ordinary differential equations has been studied by a number of authors. For the singular mixed boundary value problems for the second order differential equation, O'Regan discussed the existence theorems in [1-3]. The aim of the paper is to generalize the O'Regan results in [2-3] to the p-Laplacian ordinary operator of the form

$$\frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'} = qf(t, y, p\psi_n(y^{'})), 0 < t < 1,$$

where $\psi_n : R \to R$ is defined by $\psi_n(u) = |u|^{n-2}u$ if $u \neq 0$, and $\psi_n(0) = 0$ where $n > 1$.

We begin to discuss the two point mixed boundary value problem

$$\begin{cases} \frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'} = qf(t, y, p\psi_n(y^{'})), 0 < t < 1, \\ \lim\limits_{t\to 0^+} p^{\frac{1}{n-1}}(t)y^{'}(t) = 0, \\ ay(1) + b\lim\limits_{t\to 1^-} p^{\frac{1}{n-1}}(t)y^{'}(t) = d, a > 0, b \geq 0. \end{cases} \tag{1}$$

We first assume throughout this paper that

(H_1) $f : [0,1] \times R^2 \to R$ is continuous, $q \in (0,1)$ with $q > 0$ on (0,1);

(H_2) $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on (0,1);

(H_3) $\int_0^1 p(x)q(x)dx < \infty$ and $\int_0^1 [\frac{1}{p(s)}]^{\frac{1}{n-1}} [\int_0^s p(x)q(x)dx]^{\frac{1}{n-1}} ds < \infty$.

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LEMMA 1. Suppose (H_1) and (H_2) hold. In addition suppose $r_m(x), r(x) \in C[0,1]$ with $r_m \to r$. Then for each $\varepsilon > 0$ there is N, independent of s ($s \in [0,1]$), such that

$$|\psi_n^{-1}(\int_0^s p(x)q(x)r_m(x)dx) - \psi_n^{-1}(\int_0^s p(x)q(x)r(x)dx)| \le \varepsilon\psi_n^{-1}(\int_0^s p(x)q(x)dx)$$

when $m > N$.

PROOF. Since $r(x) \in C[0,1]$, there is constant $M^{'} > 1$ such that $|r_m(x)| \le M^{'} - 1$ for all $x \in [0,1]$. Since ψ_n^{-1} is uniformly continuous in $[-M^{'}, M^{'}]$, for every $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that $|\psi_n^{-1}(s) - \psi_n^{-1}(t)| < \varepsilon$ when $|s-t| \le \delta$ and $s, t \in [-M^{'}, M^{'}]$. Since $r_m \to r$, for the above $\delta > 0$, there is N such that $|r_m(x) - r(x)| < \delta$, for all $x \in [0,1]$ when $m > N$. From Integral Mean-value Theorem, there exists $\xi \in [0,1]$ such that

$$\int_0^s p(x)q(x)r(x)dx = r(\xi)\int_0^s p(x)q(x)dx.$$

When $m > N$, we have

$$\begin{aligned}
&|\psi_n^{-1}(\textstyle\int_0^s p(x)q(x)r_m(x)dx) - \psi_n^{-1}(\int_0^s p(x)q(x)r(x)dx)| \\
&\quad \le \max\{|\psi_n^{-1}(\textstyle\int_0^s p(x)q(x)[r(x)-\delta]dx) - \psi_n^{-1}(\int_0^s p(x)q(x)r(x)dx)|, \\
&\qquad |\psi_n^{-1}(\textstyle\int_0^s p(x)q(x)[r(x)+\delta]dx) - \psi_n^{-1}(\int_0^s p(x)q(x)r(x)dx)|\} \\
&\quad = \max\{|\psi_n^{-1}((r(\xi)-\delta)\textstyle\int_0^s p(x)q(x)dx) - \psi_n^{-1}(r(\xi)\int_0^s p(x)q(x)dx)|, \\
&\qquad |\psi_n^{-1}((r(\xi)+\delta)\textstyle\int_0^s p(x)q(x)dx) - \psi_n^{-1}(r(\xi)\int_0^s p(x)q(x)dx)|\} \\
&\quad = \max\{|\psi_n^{-1}(r(\xi)-\delta) - \psi_n^{-1}(r(\xi))|, |\psi_n^{-1}(r(\xi)+\delta) - \psi_n^{-1}(r(\xi))|\}\psi_n^{-1}(\textstyle\int_0^s p(x)q(x)dx) \\
&\quad \le \varepsilon\psi_n^{-1}(\textstyle\int_0^s p(x)q(x)dx).
\end{aligned}$$

Associated with (1), we have a family of problems

$$\begin{cases}
\frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'} = \lambda q f(t, y, p\psi_n(y^{'})), 0 < t < 1, 0 < \lambda < 1, \\
\lim\limits_{t\to 0^+} p^{\frac{1}{n-1}}(t)y^{'}(t) = 0, \\
ay(1) + b\lim\limits_{t\to 1^-} p^{\frac{1}{n-1}}(t)y^{'}(t) = d, a > 0, b \ge 0.
\end{cases} \tag{2}$$

THEOREM 1. Suppose (H_1), (H_2) and (H_3) hold and assume there is a constant M, independent of λ, with $|y|_1 \le M$ for each solution y to (2), for each $\lambda \in (0,1)$. Then (1) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $p\psi_n(y^{'}) \in C[0,1]$. Where we set $|y|_1 = \max\{|y|_0, |p\psi_n(y^{'})|_0\} = \max\{\sup_{[0,1]}|y(t)|, \sup_{[0,1]}|p(t)\psi_n(y^{'}(t))|\}$.

PROOF. Solving (2) is equivalent to finding a $y \in C[0,1]$ with $p\psi_n(y^{'}) \in C[0,1]$ which satisfies

$$\begin{aligned}
y(t) \quad &= \tfrac{d}{a} + \lambda^{\frac{1}{n-1}}[\textstyle\int_0^t (\frac{1}{p(s)})^{\frac{1}{n-1}}\psi_n^{-1}(\int_0^s p(x)q(x)f(x, y(x), p(x)\psi_n(y^{'}(x)))dx)ds \\
&\quad -\tfrac{b}{a}\psi_n^{-1}(\textstyle\int_0^1 p(x)q(x)f(x, y(x), p(x)\psi_n(y^{'}(x)))dx) \\
&\quad -\textstyle\int_0^1 \psi_n^{-1}(\frac{1}{p(s)}\int_0^s p(x)q(x)f(x, y(x), p(x)\psi_n(y^{'}(x)))dx)ds].
\end{aligned} \tag{3}$$

Let $\mu = \lambda^{\frac{1}{n-1}}$. We can write (3) as

$$y(t) = (1-\mu)\frac{d}{a} + \mu N y(t),$$

where N: $K^1_B[0,1] \to K^1_B[0,1]$ is given by

$$\begin{aligned} Ny(t) &= \tfrac{d}{a} + \int_0^t (\tfrac{1}{p(s)})^{\frac{1}{n-1}} \psi_n^{-1}(\int_0^s p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx)ds \\ &-\tfrac{b}{a}\psi_n^{-1}(\int_0^1 p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx) \\ &-\int_0^1 \psi_n^{-1}(\tfrac{1}{p(s)}\int_0^s p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx)ds. \end{aligned}$$

Let

$$K^1[0,1] = \{u \in C[0,1], p\psi_n(y') \in C[0,1] \text{ with norm } |u|_1\},$$

which is a Banach space, and set

$$K^1_B[0,1] = \{u \in K^1[0,1], \lim_{t\to 0^+} p^{\frac{1}{n-1}}(t)y'(t) = 0, au(1) + b\lim_{t\to 1^-} p^{\frac{1}{n-1}}(t)y'(t) = d\}.$$

Then $K^1_B[0,1]$ is a convex subset of $K^1[0,1]$. Consequently (2) is equivalent to the fixed point problem

$$y = (1-\mu)\frac{d}{a} + \mu Ny.$$

We first prove that N is continuous. Let y_m, $y \in K^1_B[0,1]$, and $y_m \to y$. So there is a constant M_0 so that $|y_m|_1 \le M_0$, $|y| \le M_0$. With lemma 1, for every $\varepsilon > 0$ there is N, when $m > N$, we have

$$\begin{aligned} |Ny - Ny_m|_0 &= \sup_{[0,1]} |Ny(t) - Ny_m(t)| \\ &\le 2\int_0^1 (\tfrac{1}{p(s)})^{\frac{1}{n-1}} |\psi_n^{-1}(\int_0^s p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx) \\ &-\psi_n^{-1}(\int_0^s p(x)q(x)f(x,y_m(x),p(x)\psi_n(y_m'(x)))dx)|ds \\ &+\tfrac{b}{a}|\psi_n^{-1}(\int_0^1 p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx) \\ &-\psi_n^{-1}(\int_0^1 p(x)q(x)f(x,y_m(x),p(x)\psi_n(y_m'(x)))dx)| \\ &\le 2\varepsilon\int_0^1 (\tfrac{1}{p(s)})^{\frac{1}{n-1}} \psi_n^{-1}(\int_0^s p(x)q(x)dx)ds + \tfrac{b}{a}\varepsilon, \end{aligned}$$

$$\begin{aligned} &|p\psi_n((Ny)') - p\psi_n((Ny_m)')|_0 = \sup_{[0,1]} |p(t)\psi_n((Ny)'(t)) - p(t)\psi_n(Ny_m)'(t))| \\ &\le \int_0^1 p(x)q(x)|f(x,y(x),p(x)\psi_n(y'(x))) - f(x,y_m(x),p(x)\psi_n(y_m'(x)))|dx \\ &\le \varepsilon\int_0^1 p(x)q(x)dx, \end{aligned}$$

so (H_3) and the above inequalities imply $Ny_m \to Ny$. Thus N is continuous.

Next we show N is completely continuous. Let $\Omega \subseteq K^1_B[0,1]$ be bounded i.e. there exists a constant $M_1 > 0$ with $|y|_1 \le M_1$ for each $y \in \Omega$. For each $y \in \Omega$, we have

$$\begin{aligned} |Ny|_0 &\le \tfrac{d}{a} + 2\int_0^1 (\tfrac{1}{p(s)})^{\frac{1}{n-1}} |\psi_n^{-1}(\int_0^s p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx)|ds \\ &+\tfrac{b}{a}|\psi_n^{-1}(\int_0^1 p(x)q(x)f(x,y(x),p(x)\psi_n(y'(x)))dx)|, \\ |p((Ny)')|_0 &\le \int_0^1 p(x)q(x)|f(x,y(x),p(x)\psi_n(y'(x)))|dx, \end{aligned}$$

so there is M_3 such that $|Ny|_1 \le M_3$ for each $y \in \Omega$ i.e. NΩ is completely bounded.

For each $y \in \Omega$ and $s, t \in [0,1]$ with $s < t$, we have

$$\begin{aligned} |Ny(t) - Ny(s)| &\le |\int_s^t (\tfrac{1}{p(x)})^{\frac{1}{n-1}} \psi_n^{-1}(\int_0^x p(z)q(z)f(z,y(z),p(z)\psi_n(y'(z)))dz)dx| \\ &\le M_4\int_s^t (\tfrac{1}{p(x)})^{\frac{1}{n-1}} \psi_n^{-1}(\int_0^x p(z)q(z)dz)dx, \end{aligned}$$

$$\begin{array}{rcl}|p(t)\psi_n((Ny)^{'})(t)-p(s)\psi_n((Ny)^{'})(s)| & \leq & \int_s^t p(x)q(x)|f(x,y(x),p(x)\psi_n(y^{'}(x)))|dx \\ & \leq & M_5\int_s^t p(x)q(x)dx,\end{array}$$

where M_4 and M_5 are constant. (H_3) and above inequalities imply NΩ is equicontinuous. The Arzela-Ascoli theory in [4-5] guarantees that N is completely continuous.

Let $U=\{u\in K_B^1[0,1]:|u|_1<\max\{M+1,|\frac{d}{a}|+1\}\}$, $C=K_B^1[0,1]$ and $E=K^1[0,1]$.

Set $p^*=\frac{d}{a}$, with the choice of U, $p^*\in U$ and possibility (ii) of nonlinear alternative theorem of Leray-Schauder type in [6,7] is ruled out. By the theorem, we deduce that N has a fixed point i.e. (1) has a solution $y\in C[0,1]$ with $p\psi_n(y^{'})\in C[0,1]$. The fact that $y\in C^2(0,1)$ follows from (3) with $\lambda=1$.

We consider the problem

$$\left\{\begin{array}{l}\frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'}=qf(t,y,p\psi_n(y^{'})),0<t<1,\\ \lim\limits_{t\to0^+}p^{\frac{1}{n-1}}(t)y^{'}(t)=0,\\ ay(1)+b\lim\limits_{t\to1^-}p^{\frac{1}{n-1}}(t)y^{'}(t)=0,a>0,b\geq0.\end{array}\right. \tag{4}$$

THEOREM 2. Suppose (H_1), (H_2) and (H_3) hold. In addition suppose

(H_4) there is a constant $M>0$ with $uf(t,u,0)>0$ for $|u|>M$ and $t\in[0,1]$;

(H_5) there exists $s_0<r_0$ with $s_0\leq0\leq r_0$ and $f(t,u,r_0)\leq0\leq f(t,u,s_0)$ for $t\in[0,1]$ and $u\in[-M,M]$.

Then (4) has a solution $y\in C[0,1]\cap C^2(0,1)$, with $p\psi_n(y^{'})\in C[0,1]$.

PROOF. Let y be a solution to

$$\left\{\begin{array}{l}\frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'}=\lambda qf_1(t,y,p\psi_n(y^{'})),0<t<1,0<\lambda<1,\\ \lim\limits_{t\to0^+}p^{\frac{1}{n-1}}(t)y^{'}(t)=0,\\ ay(1)+b\lim\limits_{t\to1^-}p^{\frac{1}{n-1}}(t)y^{'}(t)=0,a>0,b\geq0,\end{array}\right. \tag*{$(5)_\lambda$}$$

where

$$f_1(t,u,v)=\left\{\begin{array}{ll}f(t,u,r_0), & v\geq r_0,\\ f(t,u,v), & s_0\leq v\leq r_0,\\ f(t,u,s_0), & v\leq s_0.\end{array}\right.$$

We notice $f_1:[0,1]\times R^2\to R$ is continuous.

We first show that

$$-M\leq y(t)\leq M \text{ for } t\in[0,1]. \tag{6}$$

Suppose $|y(t)|$ achieves a positive maximum at $t_0\in[0,1]$. First if $t_0\in(0,1)$, then $y^{'}(t_0)=0$. Assume $|y(t_0)|>M$. Then (H_4) together with $s_0\leq0\leq r_0$ and the differential equation yields

$$\begin{array}{rcl}y(t_0)(p(t_0)\psi_n(y^{'}(t_0)))^{'} & = & \lambda p(t_0)q(t_0)y(t_0)f_1(t_0,y(t_0),0)\\ & = & \lambda p(t_0)q(t_0)y(t_0)f(t_0,y(t_0),0)>0,\end{array}$$

i.e. $y(t_0)(\psi_n(y^{'}(t_0)))^{'}>0$. Then one of the following conditions occurs:

(a) if $y(t_0) > 0$, then $(\psi_n(y^{'}(t_0)))^{'} > 0$. Thus there is $\delta > 0$ such that $\psi_n(y^{'}(t)) < 0$ for $t_0 - \delta < t < t_0$, i.e. $y^{'}(t) < 0$; and $\psi_n(y^{'}(t)) > 0$ for $t_0 < t < t_0 + \delta$, i.e. $y^{'}(t) > 0$, a contradiction.

(b) if $y(t_0) < 0$, then $(\psi_n(y^{'}(t_0)))^{'} < 0$. Thus there is $\delta > 0$ such that $\psi_n(y^{'}(t)) > 0$ for $t_0 - \delta < t < t_0$, i.e. $y^{'}(t) > 0$; and $\psi_n(y^{'}(t)) < 0$ for $t_0 < t < t_0 + \delta$, i.e. $y^{'}(t) < 0$, a contradiction.

Next if $t_0 = 1$, then $b > 0$. However $y(1) \lim_{t \to 1^-} p^{\frac{1}{n-1}}(t) y^{'}(t) = -\frac{a}{b} y^2(1) < 0$, i.e. there is a $\delta > 0$ such that $y(1)y^{'}(t) < 0$ for $t \in (1-\delta, 1)$, a contradiction.

It remains to consider the case $t_0 = 0$. Suppose $|y(0)| > M$, then

$$y(0) f_1(0, y(0), 0) = y(0) f(0, y(0), 0) > 0$$

together with the differential equation implies there exists $\delta > 0$ with $y(t)(p(t)\psi_n (y^{'}(t)))^{'} > 0$ for $t \in (0, \delta)$.

(c) if $y(t) > 0$ for $t \in (0, \delta)$, then $(p(t)\psi_n(y^{'}(t)))^{'} > 0$. With $\lim_{t \to 0^+} p^{\frac{1}{n-1}}(t) y^{'}(t) = 0$, we reduce $p(t)\psi_n(y^{'}(t)) > 0$ for $t \in (0, \delta)$, i.e. $y^{'}(t) > 0$, a contradiction.

(d) if $y(t) < 0$ for $t \in (0, \delta)$, then $(p(t)\psi_n(y^{'}(t)))^{'} < 0$. With $\lim_{t \to 0^+} p^{\frac{1}{n-1}}(t) y^{'}(t) = 0$, we reduce $p(t)\psi_n(y^{'}(t)) < 0$ for $t \in (0, \delta)$, i.e. $y^{'}(t) < 0$, a contradiction.

Thus(6) is true.

Next we show

$$s_0 \le p(t)\psi_n(y^{'}(t)) \le r_0 \text{ for } t \in [0, 1]. \tag{7}$$

Suppose there exists $t \in (0, 1)$ with $p(t)\psi_n(y^{'}(t)) > r_0$. Then since $\lim_{t \to 0^+} p^{\frac{1}{n-1}}(t) y^{'}(t) = 0$, $\lim_{t \to 0^+} p(t)\psi_n(y^{'}(t)) = 0$. So there exists $\mu \in [0, t)$ with $p(s)\psi_n(y^{'}(s)) > r_0$ for $s \in (\mu, t]$ and $p(\mu)\psi_n(y^{'}(\mu)) = r_0$. This together with (H_5) yields

$$\begin{aligned} 0 < p(t)\psi_n(y^{'}(t)) - p(\mu)\psi_n(y^{'}(\mu)) &= \int_\mu^t (p(s)\psi_n(y^{'}(s)))^{'} ds \\ &= \lambda \int_\mu^t p(s) q(s) f_1(s, y(s), p(s)\psi_n(y^{'}(s))) ds \\ &= \lambda \int_\mu^t p(s) q(s) f(s, y(s), r_0) ds \le 0, \end{aligned}$$

a contradiction. Thus $p(t)\psi_n(y^{'}(t)) \le r_0$ for $t \in [0, 1]$. Similarly $p(t)\psi_n(y^{'}(t)) \ge s_0$ for $t \in [0, 1]$.

Thus we have shown that any solution y of $(5)_\lambda$ satisfied (6) and (7). Consequently theorem 1 implies that $(5)_\lambda$ has solution y with $-M \le y(t) \le M, t \in [0, 1]$ and $s_0 \le p(t)\psi_n(y^{'}(t)) \le r_0, t \in [0, 1]$, so y is solution of (4) with $y \in C[0, 1] \cap C^2[0, 1]$ and $p\psi_n(y^{'}) \in C[0, 1]$.

THEOREM 3. Suppose $(H_1), (H_2), (H_3)$ and (H_4) hold. In addition assume

(H_6) there exists $s_0 < r_0$ with $s_0 \le 0 \le r_0$ and $uf(t, u, r_0) \ge 0$ for $t \in [0, 1]$ and $u \in [-M_0, M_0]$, and $uf(t, u, s_0) \ge 0$ for $t \in [0, 1]$ and $u \in [-M_0, M_0]$; here $M_0 = \max\{M, \frac{|d|}{a}\}$ and $bs_0 \le d \le br_0$.

Then (1) has a solution $y \in C[0, 1] \cap C^2[0, 1]$ and $p\psi_n(y^{'}) \in C[0, 1]$.

PROOF. Let y be a solution to

$$\begin{cases} \frac{1}{p}\left(p\psi_n\left(y^{'}\right)\right)^{'} = \lambda q f_1(t, y, p\psi_n(y^{'})), 0 < t < 1, 0 < \lambda < 1, \\ \lim\limits_{t\to 0^+} p^{\frac{1}{n-1}}(t)y^{'}(t) = 0, \\ ay(1) + b\lim\limits_{t\to 1^-} p^{\frac{1}{n-1}}(t)y^{'}(t) = d, a > 0, b \geq 0, \end{cases} \tag{8$_\lambda$}$$

where f_1 is as in theorem 2. We first show that

$$|y(t)| \leq M_0 \text{ for } t \in [0,1]. \tag{9}$$

Suppose $|y(t)|$ achieves a positive maximum at $t_0 \in [0,1]$. If $t_0 \in [0,1)$, as in Theorem 2, $|y(t_0)| \leq M$. If $t_0 = 1$, then one of the following conditions occurs:

(a) If $b = 0$, then $|y(1)| = \frac{|d|}{a} \leq M_0$.

(b) If $b \neq 0$, and suppose $|y(1)| > \frac{|d|}{a}$, then

$$by(1)\lim_{t\to 1^-} p^{\frac{1}{n-1}}(t)y^{'}(t) = y(1)d - a[y(1)]^2 \leq |y(1)||d| - a[y(1)]^2 = |y(1)|(|d| - a|y(1)|) < 0.$$

Thus there is $\delta > 0$ with $y(t)p^{\frac{1}{n-1}}(t)y^{'}(t) < 0$ for $t \in (1-\delta, 1)$, i.e. $y(t)y^{'}(t) < 0$, a contradiction. Consequently (9) is true.

Next we show

$$s_0 \leq p(t)\psi_n(y^{'}(t)) \leq r_0 \text{ for } t \in [0,1]. \tag{10}$$

Suppose $p(t)\psi_n(y^{'}(t)) \not\leq r_0$, then there is $t_1 \in (0,1)$ such that $p(t_1)\psi_n(y^{'}(t_1)) > r_0$. One of the following conditions occurs:

(c) If $y(t_1) > 0$, let $I_1 = \{t \in [t_1, 1] : p\psi_n(y^{'})(s) > r_0 \geq 0, \forall s \in [t_1, t]\}$. Then I_1 is an interval, $I_1 \neq \phi$ and $y(t) > 0$ on I_1. Suppose $1 \in I_1$. If $b = 0$, then $y_1 = \frac{d}{a} \leq 0$, a contradiction. If $b \neq 0$, then $\lim\limits_{t\to 1^-} p(t)\psi_n(y^{'}(t)) = \frac{d}{b} - \frac{a}{b}y(1) \leq \frac{d}{b} \leq r_0$, a contradiction. Thus $1 \notin I_1$. Then there is $t_1 < t_2 \in [0,1]$ such that $y(t) > 0$, $p(t)\psi_n(y^{'}(t)) > r_0$ for $t \in (t_1, t_2)$ with $p(t_2)\psi_n(y^{'}(t_2)) = r_0$.

(d) If $y(t_1) < 0$, then there is $t_0 < t_1 \in [0,1]$ such that $y(t) < 0$, $p(t)\psi_n(y^{'}(t)) > r_0$ for $t \in (t_0, t_1)$ with $p(t_0)\psi_n(y^{'}(t_0)) = r_0$.

If (c) holds, then

$$\begin{aligned} 0 > p(t_2)\psi_n(y^{'}(t_2)) - p(t_1)\psi_n(y^{'}(t_1)) &= \int_{t_1}^{t_2} (p(s)\psi_n(y^{'}(s)))^{'} ds \\ &= \lambda \int_{t_1}^{t_2} p(s)q(s)f(s, y(s), r_0) ds \geq 0, \end{aligned}$$

a contradiction.

If (d) holds, then

$$\begin{aligned} 0 < p(t_1)\psi_n(y^{'}(t_1)) - p(t_0)\psi_n(y^{'}(t_0)) &= \int_{t_0}^{t_1} (p(s)\psi_n(y^{'}(s)))^{'} ds \\ &= \lambda \int_{t_0}^{t_1} p(s)q(s)f(s, y(s), r_0) ds \geq 0, \end{aligned}$$

a contradiction.

Thus $p(t)\psi_n(y^{'}(t)) \leq r_0$ for $t \in [0,1]$. Similarly $p(t)\psi_n(y^{'}(t)) \geq s_0$ for $t \in [0,1]$, so (10) is true.

Now (9), (10) and Theorem 1 imply $(8)_\lambda$ has a solution y with $|y(t)| \leq M_0$ for $t \in [0,1]$ and $s_0 \leq p(t)\psi_n(y^{'}(t)) \leq r_0$ for $t \in [0,1]$, so y is a solution of (1).

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References

[1] D. O'Regan, Theorem of Singular Boundary Value Problems, World Scientific Publishing Co. Pte. Ltd., 1994.

[2] D. O'Regan, Some existence principles and some general results for singular nonlinear two point boundary value problems, J. Math. Anal. Appl., 166(1992), 24–40.

[3] M. Frigon and D. O'Regan, Existence results for some initial and boundary value problems without growth restriction, Proc. Amer. Math. Soc., 123(1995), 207–216.

[4] C. O. Chritenson and W. L. Voxman, Aspects of Topology, Marcel Dekker, New York, 1997.

[5] G. F. Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, Toyok, 1963.

[6] J. Dugundji and A. Ggrans, Fixed Point Theory, Monografie Matematyczne, PWN, Warsaw, 1982.

[7] A. Granas, Sur la mt́hode de cotinuitd́e Poincar, C.R. Acad. Sci. Paris, 282(1976), 983–985.