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Existence Results For Nonlinear Abstract Neutral Differential Equations With Time Varying Delays*

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Abstract

In this paper we prove the existence of mild solutions of nonlinear neutral time varying multiple delay differential equations in Banach space. The results are obtained by using the Schaefer fixed point theorem. As an application controllability problem is studied for the neutral systems.

1 Introduction

Theory of neutral differential equations has been studied by several authors in Banach spaces [3, 8, 9, 10, 11]. A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations have been discussed by Pazy [12]. Balachandran and Sakthivel [4] and Dauer and Balachandran [5] studied the existence of solutions for neutral functional integrodifferential equations in Banach spaces. Recently, with the help of Sadovskii's theorem, Fu and Ezzinbi [7] studied the existence of solutions for neutral differential equations of the form

$$\begin{aligned} \frac{d}{dt}\left[x(t) + F\left(t, x(t), x(b_1(t)), ..., x(b_m(t))\right)\right] &= Ax(t) \\ +G\left(t, x(t), x(a_1(t)), ..., x(a_n(t))\right), \; t &\in J = [0, a], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$ in a Banach space X, $F : [0, a] \times X^{m+1} \to X, G : [0, a] \times X^{n+1} \to X$ are continuous functions. The delays $a_i(t), b_j(t)$ are continuous scalar valued functions defined on J such that $a_i(t) \leq t, b_j(t) \leq t$. The purpose of this paper is to prove the existence of mild solutions for the same class of neutral equations with mild solutions by applying Schaefer's theorem instead of Sadovskii's theorem.

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2 Main Result

Let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$ defined on a Banach space X with norm $\|\cdot\|$. Let $0 \in \rho(A)$ then define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ which is dense in X. Further $D(A^\alpha)$ is a Banach space under the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \text{ for } x \in D(A^\alpha)$$

and is denoted by X_α. The imbedding $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)\}$ the following properties will be used.

(a) there is a $M_1 > 1$ such that $\|T(t)\| \leq M_1$, for all $0 \leq t \leq a$.

(b) for any $\alpha > 0$, there exists a positive constant $M_2 > 0$ such that

$$\|A^\alpha T(t)\| \leq M_2 t^{-\alpha}, 0 < t \leq a. \tag{2}$$

DEFINITION. A function $x(\cdot)$ is called a mild solution of the system (1) if $x(0) = x_0$, the restriction of $x(\cdot)$ to the interval $[0, a]$ is continuous and for each $0 \leq t \leq a$ the function $AT(t-s)F(s, x(s), x(b_1(s)), ..., x(b_m(s)))$, $s \in [0, t)$, is integrable, and the following integral equation

$$\begin{aligned} x(t) \quad = \quad & T(t)[x_0 + F(0, x_0, x(b_1(0)), ..., x(b_m(0)))] \\ & - F(t, x(t), x(b_1(t)), ..., x(b_m(t))) \\ & - \int_0^t AT(t-s)F(s, x(s), x(b_1(s)), ..., x(b_m(s)))ds \\ & + \int_0^t T(t-s)G(s, x(s), x(a_1(s)), ..., x(a_n(s)))ds \end{aligned} \tag{3}$$

is satisfied.

We need the following fixed point theorem due to Schaefer [13].

THEOREM 2.1. Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta(F) = \{x \in E : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that the following conditions hold:

(H1) For each $t \in J$, the function $G(t, \cdot) : X^{n+1} \to X$ is continuous, and for each $(x_0, x_1, ..., x_n) \in X^{n+1}$ the function $G(\cdot, x_0, x_1, ..., x_n) : [0, a] \to X$ is strongly measurable.

(H2) For each positive integer k there exists $\alpha_k \in L^1[0,a]$ such that

$$\sup_{\|x_0\|\cdots\|x_n\|\le k} \|G(t, x_0, x_1, ..., x_n)\| \le \alpha_k(t) \;\text{ for }\; t \in J.$$

(H3) The function $F : [0,a] \times X^{m+1} \to X$ is completely continuous and for any bounded set Q in $C([-r,a], X)$ the set

$$\{t \to F(t, x(t), x(a_1(t)), ..., x(a_m(t))) : x \in Q\}$$

is equicontinuous.

(H4) There exist $\beta \in (0,1)$ and a constant $c_1 \ge 0$ such that

$$\|(A)^\beta F(t, u(t))\| \le M_3, \;\; t \in J.$$

(H5) There exists an integrable function $m : [0,a] \to [0,\infty)$ such that

$$\|G(t, x(t), x(a_1(t)), ..., x(a_n(t)))\| \le (n+1)m(t)\Omega(\|x(t)\|)$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H6)

$$\int_0^a \hat{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where

$$\begin{aligned} c &= M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta}, \\ M_4 &= \|(A)^{-\alpha}\|, \\ \hat{m}(t) &= M_1 m(t)(n+1)^2. \end{aligned}$$

Now let us take

$$\begin{aligned} (t, x(t), x(b_1(t)), ..., x(b_m(t))) &= (t, u(t)), \\ (t, x(t), x(a_1(t)), ..., x(a_n(t))) &= (t, v(t)). \end{aligned}$$

THEOREM 3.1. If the above assumptions are satisfied then the problem (1) has a mild solution on $J = [0,a]$.

PROOF. Consider the Banach space $Z = C(J, X)$ with norm

$$\|x\| = \sup\{|x(t)| : t \in J\}.$$

To prove the existence of mild solution of (1) we have to apply Schaefer's theorem for the following operator equation

$$x(t) = \lambda\Psi x(t), \quad 0 < \lambda < 1, \tag{4}$$

where $\Psi : Z \to Z$ is defined as

$$
\begin{aligned}
(\Psi x)(t) &= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds \\
&\quad + \int_0^t T(t-s)G(s, v(s))ds.
\end{aligned}
$$

Then from (3) we have

$$
\begin{aligned}
\|x(t)\| &\leq M_1[\|x_0\| + M_3M_4] + M_3M_4 + M_2 \int_0^t M_3(t-s)^{\beta-1}ds \\
&\quad + M_1 \int_0^t (n+1)m(s)\Omega(\|v(s)\|)ds \\
&\leq M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta} + M_1 \int_0^t (n+1)m(s)\Omega(\|v(s)\|)ds
\end{aligned}
$$

Denoting the right hand side of above inequality as $\mu(t)$ then

$$
\|x(t)\| \leq \mu(t) \text{ and } \mu(0) = c = M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta},
$$

$$
\mu'(t) = M_1(n+1)m(t)\Omega(\|v(t)\|) \leq M_1(n+1)^2m(t)\Omega(\mu(t)) \leq \hat{m}(t)[\Omega(\mu(t))].
$$

This implies

$$
\int_{\mu(0)}^{\mu(t)} \frac{ds}{\Omega(s)} \leq \int_0^a \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq a. \tag{5}
$$

Inequality (5) implies that there is a constant K such that $\mu(t) \leq K, t \in [0, a]$ and hence we have $\|x\| = \sup\{|x(t)| : t \in J\} \leq K$, where K depends only on a and on the functions $\hat{m}$ and Ω.

We shall now prove that the operator $\Psi : Z \to Z$ is a completely continuous operator. Let $B_k = \{x \in Z : \|x\|_1 \leq k\}$ for some $k \geq 1$. We first show that Ψ maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in [0, a]$. Then if $0 < t_1 < t_2 < a$,

$$
\begin{aligned}
&\|(\Psi x)(t_1) - (\Psi x)(t_2)\| \\
&\leq \|(T(t_1) - T(t_2))[x_0 + F(0, u(0))]\| + \|F(t_1, u(t_1)) - F(t_2, u(t_2))\| \\
&\quad + \|\int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]F(s, u(s))ds\| + \|\int_{t_1}^{t_2} AT(t_2 - s)F(s, u(s))ds\| \\
&\quad + \|\int_0^{t_1} [T(t_1 - s) - T(t_2 - s)]G(s, v(s))ds\| + \|\int_{t_1}^{t_2} T(t_2 - s)G(s, v(s))ds\| \\
&\leq \|(T(t_1) - T(t_2))[x_0 + F(0, u(0))]\| + \|F(t_1, u(t_1)) - F(t_2, u(t_2))\| \\
&\quad + \int_0^{t_1} \|A[T(t_1 - s) - T(t_2 - s)]\|M_3M_4ds + \int_{t_1}^{t_2} \|AT(t_2 - s)\|M_3M_4ds \\
&\quad + \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\|\alpha_k(s)\, ds + \int_{t_1}^{t_2} \|T(t_2 - s)\|\alpha_k(s)ds.
\end{aligned}
$$

The right hand side is independent of $x \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since F is completely continuous and the compactness of $T(t)$ for $t > 0$ implies continuity in the uniform operator topology. Thus Ψ maps B_k into an equicontinuous family of functions.

It is easy to see that ΨB_k is uniformly bounded. Next, we show $\overline{\Psi B_k}$ is compact. Since we have shown ΨB_k is equicontinuous collection, by the Arzela-Ascoli theorem it suffices to show that Ψ maps B_k into a precompact set in X.

Let $0 < t \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}(\Psi_\epsilon x)(t) &= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^{t-\epsilon} AT(t-s)F(s, u(s))ds \\ &\quad + \int_0^{t-\epsilon} T(t-s)G(s, v(s))ds \\ &= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)F(s, u(s))ds \\ &\quad + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)G(s, v(s))ds\end{aligned}$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Psi_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$ we have

$$\begin{aligned}\|(\Psi x)(t) - (\Psi_\epsilon x)(t)\| &\leq \int_t^{t-\epsilon} \|AT(t-s)F(s, u(s))\|ds + \int_t^{t-\epsilon} \|T(t-s)G(s, v(s))\|ds \\ &\leq \int_t^{t-\epsilon} \|AT(t-s)F(s, u(s))\|ds + \int_t^{t-\epsilon} \|T(t-s)\|\alpha_k(s)ds.\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(\Psi x)(t) : x \in B_k\}$. Hence, the set $\{(\Psi x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $\Psi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer q such that $\|x_n(t)\| \leq q$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$. By (H2)

$$G(t, v_n(t)) \to G(t, v(t))$$

for each $t \in J$ and since

$$\|G(t, v_n(t)) - G(t, v(t))\| \leq 2\alpha_q(t),$$

we have, by the dominated convergence theorem, that

$$\begin{aligned}
\|\Psi x_n - \Psi x\| &= \sup_{t\in J} \|[F(t,u_n(t)) - F(t,u(t))] \\
&\quad + \int_0^t AT(t-s)[F(s,u_n(s)) - F(s,u(s))]ds \\
&\quad + \int_0^t T(t-s)[G(s,u_n(s)) - G(s,u(s))]ds\| \\
&\leq \|F(t,u_n(t)) - F(t,u(t))\| \\
&\quad + \int_0^t \|AT(t-s)\|\|F(s,u_n(s)) - F(s,u(s))\|ds \\
&\quad + \int_0^t \|T(t-s)\|\|G(s,u_n(s)) - G(s,u(s))\|ds \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Thus Ψ is continuous. This completes the proof that Ψ is completely continuous.

Finally the set $\zeta(\Psi) = \{x \in Z : x = \lambda\Psi x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator Ψ has a fixed point in Z. This means that any fixed point of Ψ is a mild solution of (1) on J satisfying $(\Psi x)(t) = x(t)$.

3 Application

As an application of Theorem 3.1, we shall consider the system (1) with a control parameter such as

$$\begin{aligned}
\frac{d}{dt}\left[x(t) + F\left(t, x(t), x(b_1(t),...,x(b_m(t)))\right)\right] &= Ax(t) + Bw(t) \\
+G\left(t, x(t), x(a_1(t)), \cdots\cdots, x(a_n(t))\right),\ t &\in J = [0,a], \qquad (6) \\
x(0) &= x_0,
\end{aligned}$$

where B is a bounded linear operator from U, a Banach space, to X and $w \in L^2(J,U)$.

In this case the mild solution of (6) is given by

$$\begin{aligned}
x(t) &= T(t)[x_0 + F(0,u(0))] - F(t,u(t)) - \int_0^t AT(t-s)F(s,u(s))ds \\
&\quad + \int_0^t T(t-s)[Bw(s)ds + G(s,v(s))]ds
\end{aligned}$$

We say the system (6) is locally controllable on the interval J if for any subset $Y \subset X$ and for every $x_0, x_1 \in Y$, there exists a control $w \in L^2(J,U)$ such that the solution $x(\cdot)$ of (6) satisfies $x(a) = x_1$. Let $X_r = \{x \in X : \|x\| \leq r\}$ for some $r > 0$ and $Z_r = C^1(J, X_r)$.

Controllability of nonlinear systems of various types in Banach spaces has been studied by several authors by means of fixed point principles [2]. Recently Balachandran and Anandhi [1] and Fu [6] investigated the controllability problem for neutral systems. To establish the controllability result for the system (6) we need the following additional hypotheses.

(H7) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^a T(a-s)Bw(s)ds$$

has an induced inverse operator $\tilde{W}^{-1}$ which takes values in $L^2(J, U)/\ker W$ and there exists a positive constant M_5 such that $\|B\tilde{W}^{-1}\| \leq M_5$.

(H8)

$$\int_0^a \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

where

$$\begin{aligned} c &= M_1[\|x_0\| + M_3c_1] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta} + M_1Na, \\ N &= M_5\left\{\|x_1\| + M_1(\|x_0\| + M_3M_4) + M_3M_4 + \frac{M_3M_2a^\beta}{\beta}\right. \\ &\quad \left. + M_1\int_0^a m(s)(n+1)\Omega(r)ds\right\}. \end{aligned}$$

THEOREM 4.1. If the hypotheses (H1)-(H8) are satisfied, then the system (6) is controllable.

PROOF. Using the hypotheses (H7), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} w(t) &= \tilde{W}^{-1}\{x_1 - T(a)[x_0 + F(0, u(0))] + F(a, u(a)) \\ &\quad + \int_0^a AT(a-s)F(s, u(s))ds - \int_0^a T(a-s)G(s, v(s))ds\}(t) \end{aligned}$$

We shall show that when using this control the operator $\Phi : Z_r \to Z_r$ defined by

$$\begin{aligned} (\Phi x)(t) &= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds \\ &\quad + \int_0^t T(t-s)[Bw(s) + G(s, v(s))]ds, \quad t \in J \end{aligned}$$

has a fixed point. This fixed point is then a solution of (6). Substituting $w(t)$ in the above equation we get $(\Phi x)(a) = x_1$, which means that the control w steers system (6) from the given initial condition x_0 to x_1 in time a. Thus the system (6) is controllable. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted.

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