

## Equivalent Contractive Conditions in Metric Spaces\*

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### Abstract

In this note, we introduce some contractive conditions which complete the corresponding contractive conditions appeared in references [1]-[11]. By these conditions, we prove that twenty-two contractive conditions are equivalent. Then we use these conditions to get some common fixed point theorems for noncompatible maps and some weakly compatible mappings.

There are a number of papers dealing with fixed point, common fixed points for compatible maps or noncompatible maps. About these results, one can refer to [1]-[11]. J. Jachymski divided these results into two categories. The first category assumes that the maps employed satisfy some  $(\varepsilon - \delta)$ -type conditions introduced by Meir and Keeler. The second category assumes that the maps satisfy some inequalities involving a contractive gauge function. Such a class of functions started from [2]. But many of these conditions are equivalent. Some equivalent conditions have been discussed in [3] and [4]. In this note, we introduce a limit type contractive condition (C1) and several other conditions (C2)-(C9). By these conditions, we prove that twenty-two contractive conditions are equivalent, this completes the work in [3] and [4]. These conditions were frequently used to prove existence theorems in common fixed point for compatible maps. However, the study of common fixed points of noncompatible mappings is also very interesting. In this note, by using these conditions and property (E.A) in [1], we prove some new common fixed point theorems for noncompatible maps and some weakly compatible mappings.

Let  $(X, d)$  be a metric space,  $A, B, S$  and  $T$  be selfmaps of a set  $X$ . For any  $x, y \in X$ , we define

$$M(x, y) = \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), [d(Sx, By) + d(Ty, Ax)]/2\}.$$

We list the following contractive conditions:

(C1)  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L > 0$  implies  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) < L$ , for any sequences  $\{x_n\}, \{y_n\} \subset X$ .

(C2) There exists a function  $\delta : (0, \infty) \rightarrow (0, \infty)$  such that for any  $\varepsilon > 0$ ,  $\underline{\lim}_{t \rightarrow \varepsilon} \delta(t) > \varepsilon$ , and for any  $x, y \in X$ ,  $\varepsilon \leq M(x, y) < \delta(\varepsilon)$  implies  $d(Ax, By) < \varepsilon$ .

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(C3) There exists a function  $\omega : R_+ \rightarrow R_+$  such that for any  $s > 0$ ,  $\omega(s) > s$ ,  $\underline{\lim}_{t \rightarrow s} \omega(t) > s$ , and for any  $x, y \in X$ ,  $\omega(d(Ax, By)) \leq M(x, y)$ .

(C4) There exists an increasing and continuous function  $\phi : R_+ \rightarrow R_+$  such that  $\phi(t) < t$  for  $t > 0$  and  $d(Ax, By) \leq \phi(M(x, y))$  for all  $x, y \in X$ .

(C5) There exists an increasing function  $\psi : R_+ \rightarrow R_+$ , such that  $\underline{\lim}_{s \rightarrow t^+} \psi(s) < t$  for  $t > 0$  and  $d(Ax, By) \leq \psi(M(x, y))$  for all  $x, y \in X$ .

(C6) There exists a lower semi-continuous function  $\delta : (0, \infty) \rightarrow (0, \infty)$  such that, for any  $\varepsilon > 0$ ,  $\delta(\varepsilon) > \varepsilon$  and for any  $x, y \in X$ ,  $\varepsilon \leq M(x, y) < \delta(\varepsilon)$  implies  $d(Ax, By) < \varepsilon$ .

(C7) There exists a function  $\delta : (0, \infty) \rightarrow (0, \infty)$ , such that for any  $\varepsilon > 0$ ,  $\underline{\lim}_{t \rightarrow \varepsilon^-} \delta(t) > \varepsilon$ , and for any  $x, y \in X$ ,  $\varepsilon \leq M(x, y) < \delta(\varepsilon)$  implies  $d(Ax, By) < \varepsilon$ .

(C8) There exists a function  $\omega : R_+ \rightarrow R_+$  such that  $\omega$  is non-decreasing, for any  $s > 0$ ,  $\omega(s) > s$ ,  $\underline{\lim}_{t \rightarrow s^-} \omega(t) > s$ , and for any  $x, y \in X$ ,  $\omega(d(Ax, By)) \leq M(x, y)$ .

(C9) There exists a lower semi-continuous function  $\omega : R_+ \rightarrow R_+$  such that,  $\omega(s) > s$  for  $s > 0$  and for any  $x, y \in X$ ,  $\omega[d(Ax, By)] \leq M(x, y)$ .

(C10) There exists an upper semi-continuous function  $\Phi : R_+ \rightarrow R_+$  such that  $\Phi$  is non-decreasing,  $\Phi(s) < s$  for  $s > 0$  and for any  $x, y \in X$ ,  $d(Ax, By) \leq \Phi(M(x, y))$ .

(C11) There exists a lower semi-continuous function  $\omega : R_+ \rightarrow R_+$  such that  $\omega$  is non-decreasing,  $\omega(s) > s$  for  $s > 0$  and for any  $x, y \in X$ ,  $\omega[d(Ax, By)] \leq M(x, y)$ .

(C12) There exist functions  $\beta, \eta : (0, \infty) \rightarrow (0, \infty)$  such that, for any  $\varepsilon > 0$ ,  $\beta(\varepsilon) > \varepsilon$ ,  $\eta(\varepsilon) < \varepsilon$ , and  $0 \leq M(x, y) < \beta(\varepsilon)$  imply  $d(Ax, By) < \eta(\varepsilon)$  for any  $x, y \in X$ .

(C13) There exists a function  $\delta : (0, \infty) \rightarrow (0, \infty)$  such that for any  $\varepsilon > 0$ ,  $\delta(\varepsilon) > \varepsilon$ ,  $\sup\{\delta(s) : s \in (0, \varepsilon)\} \geq \delta(\varepsilon)$  and for any  $x, y \in X$ ,  $0 \leq M(x, y) < \delta(\varepsilon)$  implies  $d(Ax, By) < \varepsilon$ .

(C14) There exists a lower semi-continuous function  $\delta : (0, \infty) \rightarrow (0, \infty)$  such that  $\delta$  is non-decreasing, for any  $\varepsilon > 0$ , and for any  $x, y \in X$ ,  $0 \leq M(x, y) < \delta(\varepsilon)$  implies  $d(Ax, By) < \varepsilon$ .

(C15) There exists a function  $\phi : R_+ \rightarrow R_+$  such that for any  $s > 0$ ,  $\phi(s) < s$ ,  $\underline{\lim}_{t \rightarrow s} \phi(t) < s$  and for any  $x, y \in X$ ,  $d(Ax, By) \leq \phi(M(x, y))$ .

(C16) There exists an increasing and right continuous function  $\phi : R_+ \rightarrow R_+$  such that  $\phi(t) < t$  for  $t > 0$  and  $d(Ax, By) \leq \phi(M(x, y))$  for all  $x, y \in X$ .

(C17) There exists a continuous function  $\psi : R_+ \rightarrow R_+$  with  $\psi(t) > 0$  for  $t > 0$ , such that  $d(Ax, By) \leq M(x, y) - \psi(M(x, y))$  for all  $x, y \in X$ .

(C18) There exists an upper semi-continuous  $\phi : R_+ \rightarrow R_+$  such that  $\phi(t) < t$  for  $t > 0$  and  $d(Ax, By) \leq \phi(M(x, y))$  for all  $x, y \in X$ .

(C19) There exists a strictly increasing and continuous function  $\phi : R_+ \rightarrow R_+$  such that  $\phi(t) < t$  for  $t > 0$  and  $d(Ax, By) \leq \phi(M(x, y))$  for all  $x, y \in X$ .

(C20) There exists a map  $\theta : X \times X \rightarrow R_+$  with  $\inf\{\theta(x, y) : a \leq M(x, y) \leq b\} > 0$  for  $a, b > 0$ , such that  $d(Ax, By) \leq M(x, y) - \theta(x, y)$ , for all  $x, y \in X$ .

(C21) There exists a map  $\Gamma : X \times X \rightarrow R_+$  with  $\sup\{\Gamma(x, y) : a \leq M(x, y) \leq b\} < 1$  for  $a, b > 0$ , such that  $d(Ax, By) \leq \Gamma(x, y)M(x, y)$ , for all  $x, y \in X$ .

(C22) There exists a strictly increasing function  $\phi : R_+ \rightarrow R_+$  such that  $\phi(t) < t$  for  $t > 0$ ,  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ , for all  $t \in R_+$  and  $d(Ax, By) \leq \phi(M(x, y))$  for all  $x, y \in X$ .

LEMMA 1. (C10)-(C14) are equivalent.

Indeed, let  $Q = \{(M(x, y), d(Ax, By)) : x, y \in X\}$ . By the lemma in [3], we know that (C10)-(C14) are equivalent.

LEMMA 2. (C15)-(C22) are equivalent.

The proof is similar to that of Theorem 1 in [4]. We only need to replace  $d(Tx, Ty)$  with  $d(Ax, By)$  and  $d(x, y)$  with  $M(x, y)$ . Then we can prove that (C15)-(C22) are equivalent.

LEMMA 3. (C10)-(C22) are equivalent.

Since (C10) $\Rightarrow$ (C17) and (C19) $\Rightarrow$ (C10) are obvious. From Lemma 1 and Lemma 2 we know that (C10)-(C22) are equivalent.

THEOREM 1. (C1)-(C22) are equivalent.

PROOF. (C6) $\Rightarrow$ (C2) $\Rightarrow$ (C7) is obvious. Assume that (C7) holds, and  $\{x_n\}, \{y_n\} \subset X$ ,  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L > 0$ . Since  $\underline{\lim}_{t \rightarrow L^-} \delta(t) > L$ , then there exists  $\eta > 0$ , such that  $\delta(L - \eta) > L$ . It follows from  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L$ , there are infinitely many  $n$  such that  $L - \eta \leq M(x_n, y_n) < \delta(L - \eta)$ . Thus (C7) implies  $d(Ax_n, By_n) \leq L - \eta$  for such  $n$ . This implies  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) \leq L - \eta < L$ . Therefore, (C1) holds.

(C1) $\Rightarrow$ (C12). Indeed, suppose that (C1) holds, to prove (C12), it is enough to prove that for any  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  and  $0 < \eta(\varepsilon) < \varepsilon$  such that for any  $x, y \in X$ ,

$$M(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies that } d(Ax, By) < \eta(\varepsilon).$$

If this is not so, then there exist an  $\varepsilon_0 > 0$  and sequences  $\{x_n\}, \{y_n\}$  such that

$$M(x_n, y_n) < \varepsilon_0 + \frac{1}{n} \text{ and } d(Ax_n, By_n) \geq \varepsilon_0 - \frac{1}{n}. \tag{1}$$

Then  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L \leq \varepsilon_0$  and  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) \geq \varepsilon_0$ . Hence, by (C1) we infer that  $L = 0$ , that is,  $\lim_{n \rightarrow \infty} M(x_n, y_n) = L = 0$ . Suppose that  $M(x_n, y_n) > 0$  for infinitely many  $n \in N$ . By passing to a subsequence if necessary, we may assume that this inequality holds for all  $n \in N$ . Since (C1) implies that  $d(Ax, By) < M(x, y)$  if  $M(x, y) > 0$ , we may infer that  $\lim_{n \rightarrow \infty} d(Ax_n, By_n) = 0$ . But by (1) we have  $\lim_{n \rightarrow \infty} d(Ax_n, By_n) \geq \varepsilon_0$ , a contradiction. Therefore we conclude that  $M(x_n, y_n) = 0$  for  $n$  large enough. Then  $d(Ax_n, By_n) = 0$  for all such  $n$ , which contradicts with (1). Thus, (C12) holds.

Since (C14) $\Rightarrow$ (C6) is obvious, by using Lemma 3 we can get that (C1), (C2), (C6), (C7) and (C10)-(C22) are equivalent.

(C11) $\Rightarrow$ (C9) $\Rightarrow$ (C3) $\Rightarrow$ (C8) is obvious.

(C8) $\Rightarrow$ (C1). Assume that (C8) holds, and  $\{x_n\}, \{y_n\} \subset X$ ,  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L > 0$ , then

$$\underline{\lim}_{n \rightarrow \infty} \omega(d(Ax_n, By_n)) \leq \underline{\lim}_{n \rightarrow \infty} M(x_n, y_n) \leq \overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L.$$

If  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) = 0$ , then  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) < L$ . If  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) = r > 0$ , then there exists a subsequence  $\{d(Ax_{n_i}, By_{n_i})\}$  such that  $\lim_{i \rightarrow \infty} d(Ax_{n_i}, By_{n_i}) = r$ . If  $0 < d(Ax_{n_i}, By_{n_i}) \leq r$  for infinitely  $i \in N$ , then

$$\begin{aligned} \underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) &= \lim_{i \rightarrow \infty} d(Ax_{n_i}, By_{n_i}) = r < \underline{\lim}_{t \rightarrow r^-} \omega(r) \\ &\leq \underline{\lim}_{i \rightarrow \infty} \omega(d(Ax_{n_i}, By_{n_i})) \leq \overline{\lim}_{i \rightarrow \infty} M(x_{n_i}, y_{n_i}) \leq L. \end{aligned}$$

If there is an  $i_0$ , such that for any  $i \geq i_0$ ,  $d(Ax_{n_i}, By_{n_i}) > r$ , then

$$\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) = r < \omega(r) \leq \omega(d(Ax_{n_i}, By_{n_i})) \leq M(x_{n_i}, y_{n_i}),$$

for any  $i \geq i_0$ . Thus,  $\underline{\lim}_{n \rightarrow \infty} d(Ax_n, By_n) = r < \omega(r) \leq \overline{\lim}_{i \rightarrow \infty} M(x_{n_i}, y_{n_i}) \leq L$ . That is (C1) holds. Therefore, we have proved that (C1), (C3), (C8), (C9) and (C10)-(C22) are equivalent.

(C4) $\Rightarrow$ (C15), (C19) $\Rightarrow$ (C4) and (C19) $\Rightarrow$ (C5) $\Rightarrow$ (C15) are clear. From Lemma 3 we know that (C4), (C5) and (C10)-(C22) are equivalent. So we have proved that (C1)-(C22) are equivalent. The proof is completed.

DEFINITION 1. Let  $A, B, S$  and  $T$  be selfmaps of a set  $X$  such that  $A(X) \subseteq T(X)$  and  $B(X) \subseteq S(X)$ . For any  $x_0 \in X$ , any sequence  $\{y_n\}$  defined by  $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$  and  $y_{2n} = Sx_{2n} = Bx_{2n-1}$  for  $n \in N$ , the positive integers, is called an  $S, T$ -iteration of  $x_0$  under  $A$  and  $B$ .

Let us recall that two selfmaps  $A$  and  $S$  of a complete metric space  $(X, d)$  are said to be compatible [8] if  $\lim_{n \rightarrow \infty} d(ASx_n, SAx_n) = 0$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = u$  for some  $u \in X$ .

REMARK 1. From Theorem 3.3 in [5] and Theorem 1, we may see that if  $A, B, S$ , and  $T$  are selfmappings of a metric space  $(X, d)$  satisfying

(H1)  $A$  is compatible with  $S$  and  $B$  is compatible with  $T$ ;

(H2)  $AX \subset TX$  and  $BX \subset SX$

and one of (C1)-(C22). If one of  $A, B, S$  and  $T$  is continuous, then  $A, B, S$  and  $T$  have a unique common fixed point, and for any  $x_0 \in X$ , each  $S, T$ -iteration of  $x_0$  under  $A$  and  $B$  converges to the unique common fixed point.

We recall that two selfmappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to be weakly compatible, defined by Jungck, if they commute at their coincidence points; i.e. if  $Tu = Su$  for some  $u \in X$ , then  $TSu = STu$ .

DEFINITION 2 [1]. Let  $S$  and  $T$  be two selfmappings of a metric space  $(X, d)$ . We say that  $S$  and  $T$  satisfy the property  $(E.A)$  if there exists a sequence  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = u$  for some  $u \in X$ .

From Remark 1 in [1] we know that two noncompatible selfmappings of a metric space  $(X, d)$  satisfy  $(E.A)$ . In the following we will prove some common fixed point theorems for noncompatible mappings.

THEOREM 2. Let  $A, B, S$ , and  $T$  be selfmappings of a metric space  $(X, d)$  such that

(1) (C1) holds;

(2)  $A, S$  are weakly compatible and  $B, T$  are weakly compatible;

(3)  $A, S$  satisfy the property  $(E.A)$  or  $B, T$  satisfy the property  $(E.A)$ ;

(4)  $AX \subset TX$  and  $BX \subset SX$ .

If one of the range of the mappings  $A, B, S$  or  $T$  is a closed subspace of  $X$ , then  $A, B, S$ , and  $T$  have a unique common fixed point.

PROOF. Suppose that  $B, T$  satisfy the property  $(E.A)$ . Then there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = t$ , for some  $t \in X$ . Since  $BX \subset SX$ , there exists a sequence  $\{y_n\} \subset X$  such that  $Bx_n = Sy_n$ . Hence  $\lim_{n \rightarrow \infty} Sy_n = t$ .

Let us show that  $\lim_{n \rightarrow \infty} Ay_n = t$ . Since  $d(Ay_n, t) \leq d(Ay_n, Bx_n) + d(Bx_n, t)$ , it is enough to prove that  $\lim_{n \rightarrow \infty} d(Ay_n, Bx_n) = 0$ . If not, then  $\overline{\lim}_{n \rightarrow \infty} d(Ay_n, Bx_n) = r > 0$ . So, there exists a subsequence  $\{n_k\}$  such that  $\lim_{k \rightarrow \infty} d(Ay_{n_k}, Bx_{n_k}) = r$ . Since

$$\begin{aligned} M(y_{n_k}, x_{n_k}) &= \max\{d(Sy_{n_k}, Tx_{n_k}), d(Sy_{n_k}, Ay_{n_k}), d(Tx_{n_k}, Bx_{n_k}), \\ &\quad [d(Sy_{n_k}, Bx_{n_k}) + d(Tx_{n_k}, Ay_{n_k})]/2\} \\ &= \max\{d(Bx_{n_k}, Tx_{n_k}), d(Bx_{n_k}, Ay_{n_k}), d(Tx_{n_k}, Bx_{n_k}), \\ &\quad [d(Bx_{n_k}, Bx_{n_k}) + d(Tx_{n_k}, Ay_{n_k})]/2\}, \end{aligned}$$

we have that  $\lim_{k \rightarrow \infty} M(y_{n_k}, x_{n_k}) = \lim_{k \rightarrow \infty} d(Ay_{n_k}, Bx_{n_k}) = r > 0$ , this contradicts with (C1). Therefore,  $\lim_{n \rightarrow \infty} d(Ay_n, Bx_n) = 0$ , and then  $\lim_{n \rightarrow \infty} Ay_n = t$ .

Suppose that  $SX$  is a closed subspace of  $X$ . Then  $t = Su$  for some  $u \in X$ . Subsequently, we have  $\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = Su$ . Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} M(u, x_n) &= \lim_{n \rightarrow \infty} \max\{d(Su, Tx_n), d(Su, Au), d(Tx_n, Bx_n), \\ &\quad [d(Su, Bx_n) + d(Tx_n, Au)]/2\} \\ &= d(Su, Au) = \lim_{n \rightarrow \infty} d(Au, Bx_n). \end{aligned}$$

This and (C1) imply that  $d(Su, Au) = 0$ , that is,  $Su = Au$ . The weak compatibility of  $A$  and  $S$  implies that  $ASu = SAu$  and then  $AAu = ASu = SAu = SSAu$ .

On the other hand, since  $AX \subset TX$ , there exists  $v \in X$  such that  $Au = Tv$ . Since

$$\begin{aligned} M(u, v) &= \max\{d(Su, Tv), d(Su, Au), d(Tv, Bv), [d(Su, Bv) + d(Tv, Au)]/2\} \\ &= d(Au, Bv), \end{aligned}$$

by (C1) we get that  $d(Au, Bv) = 0$ , which implies that  $Au = Su = Tv = Bv$ . The weak compatibility of  $B$  and  $T$  implies that  $BTv = TBv$  and  $TTv = TBv = BTv = BBv$ .

Let us show that  $Au$  is a common fixed point of  $A, B, T$  and  $S$ . Since

$$\begin{aligned} M(Au, v) &= \max\{d(SAu, Tv), d(SAu, AAu), d(Tv, Bv), \\ &\quad [d(SAu, Bv) + d(Tv, AAu)]/2\} \\ &= d(AAu, Bv), \end{aligned}$$

by (C1) we get that  $d(AAu, Bv) = 0$ . Therefore,  $AAu = SAu = Bv = Au$  and  $Au$  is a common fixed point of  $A$  and  $S$ . Similarly, we can prove that  $Bv$  is a common fixed point of  $B$  and  $T$ . Since  $Au = Bv$ , we conclude that  $Au$  is a common fixed point  $A, B, T$  and  $S$ . The proof is similar when  $TX$  is assumed to be a closed subspace of  $X$ . The cases in which  $AX$  or  $BX$  is a closed subspace of  $X$  are similar to the cases in which  $TX$  or  $SX$ , respectively, is closed since  $AX \subset TX$  and  $BX \subset SX$ . If  $Au = Bu = Tu = Su = u$  and  $Av = Bv = Tv = Su = v$ , then

$$\begin{aligned} M(u, v) &= \max\{d(Su, Tv), d(Su, Au), d(Tv, Bv), [d(Su, Bv) + d(Tv, Au)]/2\} \\ &= d(u, v) = d(Au, Bv). \end{aligned}$$

Therefore, by (C1) we have  $u = v$ , that is, the common fixed point is unique. The proof is completed.

From Theorem 1 and Theorem 2 we have the following Theorem 3.

**THEOREM 3.** Let  $A, B, S,$  and  $T$  be selfmappings of a metric space  $(X, d)$  such that

- (1) One of (C1)-(C22) holds;
- (2)  $A, S$  are weakly compatible and  $B, T$  are weakly compatible;
- (3)  $A, S$  satisfy the property (E.A) or  $B, T$  satisfy the property (E.A);
- (4)  $AX \subset TX$  and  $BX \subset SX$ .

If one of the range of the mappings  $A, B, S,$  or  $T$  is a closed subspace of  $X$ , then  $A, B, S,$  and  $T$  have a unique common fixed point.

**THEOREM 4.** Let  $\{A_n\}_{n \in N}, \{B_n\}_{n \in N}$  be sequences of selfmaps of a metric space  $(X, d)$  such that for any  $n \in N$ ,

- (1)  $A_{2n-1}, A_{2n}, B_{2n-1}$  and  $B_{2n}$  satisfy one of (C1)-(C22);
- (2)  $A_{2n-1}, B_{2n-1}$  are weakly compatible and  $A_{2n}, B_{2n}$  are weakly compatible;
- (3)  $A_{2n-1}, B_{2n-1}$  satisfy the property (E.A) or  $A_{2n}, B_{2n}$  satisfy the property (E.A);
- (4)  $A_{2n-1}X \subset B_{2n}X, A_{2n}X \subset B_{2n-1}X$ ;
- (5) one of the range of the mappings  $A_{2n-1}, A_{2n}, B_{2n-1}$  and  $B_{2n}$  is a closed subspace of  $X$ .

Then  $\{A_n\}_{n \in N}, \{B_n\}_{n \in N}$  has a unique common fixed point.

**PROOF.** Assume that (C1) is satisfied. From Theorem 3, for any  $n \in N$ ,  $A_{2n-1}, A_{2n}, B_{2n-1}$  and  $B_{2n}$  have a unique common fixed point  $x_n$   $n = 1, 2, \dots$ . This shows that

$$d(A_{2n}x_n, A_{2n+1}x_{n+1}) = d(x_n, x_{n+1}),$$

and

$$\begin{aligned} & M_{2n, 2n+1}(x_n, x_{n+1}) \\ &= \max\{d(B_{2n}x_n, B_{2n+1}x_{n+1}), d(B_{2n}x_n, A_{2n}x_n), \\ & \quad d(B_{2n+1}x_{n+1}, A_{2n+1}x_{n+1}), [d(B_{2n}x_n, A_{2n+1}x_{n+1}) + d(B_{2n+1}x_{n+1}, A_{2n}x_n)]/2\} \\ &= d(x_n, x_{n+1}). \end{aligned}$$

From (C1), we have  $d(x_n, x_{n+1}) = 0$ , that is  $x_n = x_{n+1}$  for  $n = 1, 2, \dots$ . Therefore,  $z = x_n = x_{n+1}$ ,  $n = 1, 2, \dots$ , is the unique common fixed point of  $\{A_n\}_{n \in N}$  and  $\{B_n\}_{n \in N}$ . The proof is completed.

**THEOREM 5.** Let  $(X, d)$  be a metric space and  $S, T$  be selfmaps of  $X$ . Suppose that a sequence  $\{A_n\}_{n \in N}$  of selfmaps of  $X$ , such that for any  $n \in N$ ,

- (1)  $A_{2n-1}, A_{2n}, S$  and  $T$  satisfy one of (C1)-(C22);
- (2)  $A_{2n-1}, S$  are weakly compatible and  $A_{2n}, T$  are weakly compatible;
- (3)  $A_{2n-1}, S$  satisfy the property (E.A) or  $A_{2n}, T$  satisfy the property (E.A);
- (4)  $A_{2n-1}X \subset TX, A_{2n}X \subset SX$ ;
- (5) one of the range of the mappings  $A_{2n-1}, A_{2n}, S$  and  $T$  is a closed subspace of  $X$ .

Then  $\{A_n\}_{n \in N}$  has a unique common fixed point.

Indeed, let  $B_{2n-1} = S$  and  $B_{2n} = T$  for  $n = 1, 2, \dots$ . The conclusion can then be deduced from Theorem 4.

Now we give an example to support our result.

EXAMPLE 1. Let  $X = [1, +\infty)$  with the usual metric  $d(x, y) = |x - y|$ . Define  $A = B, T = S : X \rightarrow X$  by  $Ax = 2x - 1$  and  $Tx = x^2, \forall x \in X$ . Then

- (1)  $A$  and  $T$  satisfy the property (E.A) for the sequence  $x_n = 1 + 1/n, n = 1, 2, \dots$ .
- (2)  $A$  and  $T$  are weakly compatible.
- (3)  $AX \subset TX$ .
- (4) (C1) holds. In fact, if  $\overline{\lim}_{n \rightarrow \infty} M(x_n, y_n) = L > 0$ , then from

$$M(x_n, y_n) = \max\{|x_n - y_n| |x_n + y_n|, |x_n - 1|^2, |y_n - 1|^2, [|x_n^2 - 2y_n + 1| + |y_n^2 - 2x_n + 1|]/2\},$$

we have that  $\{x_n\}$  and  $\{y_n\}$  are bounded sequences. Thus, without loss of generality, we can assume that  $\lim_{n \rightarrow \infty} x_n = x_0, \lim_{n \rightarrow \infty} y_n = y_0$  and  $\lim_{n \rightarrow \infty} M(x_n, y_n) = L$ . This implies that

$$L = \max\{|x_0 - y_0| |x_0 + y_0|, |x_0 - 1|^2, |y_0 - 1|^2, [|x_0^2 - 2y_0 + 1| + |y_0^2 - 2x_0 + 1|]/2\},$$

and  $\lim_{n \rightarrow \infty} d(Ax_n, Ay_n) = 2|x_0 - y_0| < |x_0 - y_0| |x_0 + y_0| \leq L$ .

- (5)  $AX = TX = [1, +\infty)$  is closed.
- (6)  $A$  and  $T$  have a unique common fixed point  $x = 1$ .

EXAMPLE 2. Let  $X = [0, 1]$  with the usual metric  $d(x, y) = |x - y|$ . Define  $A = B, T = S : X \rightarrow X$  by

$$Ax = \begin{cases} 1/2, & \text{if } x = 0; \\ 0, & \text{if } x = 1; \\ x/2, & \text{if } x \in (0, 1). \end{cases}$$

and  $Tx = x, \forall x \in X$ . Then

- (1)  $A$  and  $T$  satisfy the property (E.A) for the sequence  $x_n = 1/n, n = 1, 2, \dots$ .
- (2)  $A$  and  $T$  are weakly compatible.
- (3)  $AX \subset TX$ .
- (4) (C1) does not hold. In fact, for  $x_n = 1/n$  and  $y_n = 0$ , then

$$M(1/n, 0) = \max\{1/n, 1/2n, 1/2, [|1/n - 1/2| + 1/2n]/2\} \rightarrow 1/2,$$

and  $\lim_{n \rightarrow \infty} d(A\frac{1}{n}, A0) = 1/2$ . Thus, (C1) does not hold.

- (5)  $AX = [0, 1], TX = [0, 1]$  are closed.
- (6)  $A$  and  $T$  have no common fixed point.

Example 2 also shows that the condition (C1) is important for the existence of common fixed point.

From Theorem 3, we can prove the following common fixed point theorems for weakly compatible mappings.

THEOREM 6. Let  $A, B, S,$  and  $T$  be selfmappings of a complete metric space  $(X, d)$  such that

- (1) One of (C1)-(C22) holds;
- (2)  $A, S$  are weakly compatible and  $B, T$  are weakly compatible;

(3)  $AX \subset TX$  and  $BX \subset SX$ .

If one of the range of the mappings  $A, B, S$ , or  $T$  is a closed subspace of  $X$ , then  $A, B, S$ , and  $T$  have a unique common fixed point.

PROOF. Fix  $x_0 \in X$ , from Corollary 3.2 in [5], the  $S, T$ -iteration of  $x_0$  under  $A$  and  $B$  is a Cauchy sequence. Since  $X$  is a complete metric space, the  $S, T$ -iteration sequence of  $x_0$  is a convergent sequence. Thus, we have  $\lim_{n \rightarrow \infty} Bx_{2n-1} = \lim_{n \rightarrow \infty} Tx_{2n-1}$ , and  $\lim_{n \rightarrow \infty} Ax_{2n-2} = \lim_{n \rightarrow \infty} Sx_{2n}$ . This shows that  $A, S$  satisfy the property (E.A) and  $B, T$  satisfy the property (E.A). Then from Theorem 3 we know that  $A, B, S$ , and  $T$  have a unique common fixed point. The proof is completed.

REMARK 2. Theorem 6 can be looked as an complement for common fixed point theorems of compatible mappings, such as Theorem 3.3 in [5], which is a general common fixed point theorem for compatible mappings.

Similar to Theorem 4 and Theorem 5, from Theorem 6 we can get the following theorems.

THEOREM 7. Let  $\{A_n\}_{n \in N}, \{B_n\}_{n \in N}$  be sequences of selfmaps of a complete metric space  $(X, d)$  such that for any  $n \in N$ ,

- (1)  $A_{2n-1}, A_{2n}, B_{2n-1}$  and  $B_{2n}$  satisfy one of (C1)-(C22);
- (2)  $A_{2n-1}, B_{2n-1}$  are weakly compatible and  $A_{2n}, B_{2n}$  are weakly compatible;
- (3)  $A_{2n-1}X \subset B_{2n}X, A_{2n}X \subset B_{2n-1}X$ ;
- (4) one of the range of the mappings  $A_{2n-1}, A_{2n}, B_{2n-1}$  and  $B_{2n}$  is a closed subspace of  $X$ .

Then  $\{A_n\}_{n \in N}, \{B_n\}_{n \in N}$  has a unique common fixed point.

THEOREM 8. Let  $(X, d)$  be a complete metric space and  $S, T$  be selfmaps of  $X$ . Suppose that a sequence  $\{A_n\}_{n \in N}$  of selfmaps of  $X$ , such that for any  $n \in N$ ,

- (1)  $A_{2n-1}, A_{2n}, S$  and  $T$  satisfy one of (C1)-(C22);
- (2)  $A_{2n-1}, S$  are weakly compatible and  $A_{2n}, T$  are weakly compatible;
- (3)  $A_{2n-1}X \subset TX, A_{2n}X \subset SX$ ;
- (4) one of the range of the mappings  $A_{2n-1}, A_{2n}, S$  and  $T$  is a closed subspace of  $X$ .

Then  $\{A_n\}_{n \in N}$  has a unique common fixed point.

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