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Monotone Technique For Second Order Discontinuous Differential Inclusions*

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Abstract

In this paper, two existence theorems for second order ordinary differential inclusions are proved without the continuity of multi-valued functions involved in the inclusions and using the multi-valued fixed point theorem of Dhage [4].

1 Introduction

Let $\mathbb{R}$ be a real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Consider the second order differential inclusion (in short DI)

$$\left.\begin{aligned} x''(t) &\in F(t, x(t)) \text{ a.e. } t \in J \\ x^{(i)}(0) &= x_i \in \mathbb{R},\ i = 0, 1; \end{aligned}\right\} \tag{1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ and $\mathcal{P}_p(\mathbb{R})$ is a class of all non-empty subsets of $\mathbb{R}$ with property p.

By a solution of the DI (1) we mean a function $x \in AC^1(J, \mathbb{R})$ that satisfies $x''(t) = v(t)$ for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(t))$ a. e. $t \in J$, and $x^{(i)}(0) = x_i \in \mathbb{R}$, $i = 0, 1;$ where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.

The DI (1) has already been studied in the literature for the existence results under different continuity conditions of F. The existence theorem for DI (1) for upper semi-continuous multi-valued function F is proved in Benchohra [2]. When F has closed convex values and is lower semi-continuous, the existence results of DI (1) reduce to existence results of ordinary second order differential equations

$$\left.\begin{aligned} x''(t) &= f(t, x(t)) \text{ a. e. } t \in J \\ x^{(i)}(0) &= x_i \in \mathbb{R},\ i = 0, 1; \end{aligned}\right\} \tag{2}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t, x(t)) \in F(t, x(t))$ a.e. $t \in J$.

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The case of discontinuous multi-valued function F has been treated in Dhage et. al. [6] under monotonic conditions of F and proved the existence of extremal solutions using a lattice fixed point theorem Dhage and Regan [7] in complete lattices. Note that the monotonic condition used in the above paper is of very strong nature and not every Banach space is a complete lattice. These facts motivated us to pursue the study of the present paper. In this paper we prove the existence results for the DI (1) under a monotonic condition which is weaker than that presented in Dhage et. al. [6].

2 Auxiliary Results

We equip the function space $C(J,\mathbb{R})$ with the supremum norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Clearly $C(J,\mathbb{R})$ is a Banach space with this supremum norm. Define an order relation $\leq$ in $C(J,\mathbb{R})$ by

$$x \leq y \iff x(t) \leq y(t) \quad \forall\, t \in J.$$

Then $C(J,\mathbb{R})$ is now becomes an ordered Banach space with respect to the above order relation in it.

Let X be an ordered Banach space and let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean " for every $a \in A$ there a $b \in B$ such that $a \leq b$ ". Again, $A \overset{d}{\leq} B$ means for each $b \in B$ there exists a $a \in A$ such that $a \leq b$. Further, we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. See Dhage [5] and the references therein for the details.

DEFINITION 1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp. $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ with $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X. Finally, Q is called strictly monotone increasing if $Qx \leq Qy$ for all $x, y \in X$ for which $x < y$.

We need the following fixed point theorem of Dhage [4] in the sequel.

THEOREM 1. Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in $[a,b]$, then Q has a fixed point.

3 Existence Results

We need the following definitions in the sequel.

DEFINITION 2. A multi-valued map $F : J \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, F(t)) = \inf\{|y - x| : x \in F(t)\}$ is measurable.

DEFINITION 3. A multi-valued function $F(t,x)$ is called right monotone increasing in x almost everywhere for $t \in J$ if $F(t,x) \overset{i}{\leq} F(t,y)$ a. e. $t \in J$, for all $x, y \in \mathbb{R}$ with $x \leq y$.

DEFINITION 4. A multi-valued function $\beta : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \mapsto \beta(t,x)$ is right monotone increasing almost everywhere for $t \in J$, and

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$\|\beta(t,x)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t,x)\} \leq h_r(t) \text{ a. e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Denote

$$S^1_F(x) = \{v \in L^1(J, \mathbb{R}) \mid v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}$$

for some $x \in C(J, \mathbb{R})$. The integral of the multi-valued function F is defined as

$$\int_0^t F(s, x(s))\, ds = \left\{ \int_0^t v(s)\, ds : v \in S^1_F(x) \right\}.$$

DEFINITION 5. A function $a \in AC^1(J, \mathbb{R})$ is called a lower solution of the DI (1) if for all $v \in S^1_F(a)$,

$$a''(t) \leq v(t) \text{ a.e. } t \in J$$
$$a(0) \leq x_0,\ a'(0) \leq x_1.$$

Similarly an upper solution b to DI (1) is defined.

We consider the following set of hypotheses in the sequel.

(H_1) $F(t,x)$ is closed and bounded for each $t \in J$ and $x \in \mathbb{R}$.

(H_2) $S^1_F(x) \neq \emptyset$ and the map $x \mapsto S^1_F(x)$ is right monotone increasing in $x \in \mathbb{R}$.

(H_3) F is L^1-Chandrabhan.

(H_4) DI (1) has a lower solution a and an upper solution b with $a \leq b$.

Hypotheses $(H_1) - (H_2)$ are common in the literature. Some nice sufficient conditions for guarantying (H_2) appear in Deimling [3], and Lasota and Opial [9]. A mild hypothesis of (H_4) has been used in Halidias and Papageorgiou [8]. Hypothesis (H_3) relatively new to the literature, but the special forms have been appeared in the works of several authors. See Dhage [4, 5] and the references therein for the details.

THEOREM 2. Assume that $(H_1)-(H_4)$ hold. Then the DI (1) has a solution in $[a,b]$ defined on J.

PROOF. Let $X=C(J,\mathbb{R})$ and let $Y=AC^1(J,\mathbb{R})\subset C(J,\mathbb{R})=X$. Define an order interval $[a,b]$ in Y which is well defined in view of hypothesis (H_4). Now the DI (1) is equivalent to the integral inclusion

$$x(t)\in x_0+x_1t+\int_0^t (t-s)F(s,x(s))\,ds,\ t\in J. \tag{3}$$

See Dhage et. al. [6] and the references therein. Define a multi-valued operator $Q:[a,b]\to\mathcal{P}_p(X)$ by

$$\begin{aligned} Qx&=\left\{u\in X: u(t)=x_0+x_1t+\int_0^t (t-s)v(s)\,ds,\ v\in S_F^1(x)\right\}\\ &=(\mathcal{L}\circ S_F^1)(x)\end{aligned} \tag{4}$$

where $\mathcal{L}:L^1(J,\mathbb{R})\to C(J,\mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}x(t)=x_0+x_1t+\int_0^t (t-s)x(s)\,ds. \tag{5}$$

Clearly the operator Q is well defined in view of hypothesis (H_2). We shall show that Q satisfies all the conditions of Theorem 1.

Step I : First, we show that Q has compact values on $[a,b]$. Observe that the operator Q is equivalent to the composition $\mathcal{L}\circ S_F^1$ of two operators on $L^1(J,\mathbb{R})$, where $\mathcal{L}:L^1(J,\mathbb{R})\to X$ is the continuous operator defined by (5). To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{L}\circ S_F^1$ has compact values on $[a,b]$. Let $x\in[a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(x)$. Then, by the definition of S_F^1, $v_n(t)\in F(t,x(t))$ a. e. for $t\in J$. Since $F(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t)\in F(t,x(t))$ a.e. for $t\in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t)\to\mathcal{L}v(t)$ pointwise on J as $n\to\infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t,\,\tau\in J$; then

$$\begin{aligned}|\mathcal{L}v_n(t)-\mathcal{L}v_n(\tau)|&\le|x_1|\,|t-\tau|+\left|\int_0^t (t-s)v_n(s)\,ds-\int_0^\tau(\tau-s)v_n(s)\,ds\right|\\ &\le|x_1||t-\tau|+\Big|\int_0^t (t-s)v_n(s)\,ds-\int_0^t(\tau-s)v_n(s)\,ds\Big|\\ &\quad+\Big|\int_0^t (\tau-s)v_n(s)\,ds-\int_0^\tau(\tau-s)v_n(s)\,ds\Big|\\ &\le|x_1||t-\tau|+\Big|\int_0^t (t-\tau)v_n(s)\,ds\Big|+\Big|\int_\tau^t(\tau-s)v_n(s)\,ds\Big|\\ &\le|x_1|\,|t-\tau|+\Big|\int_0^T |t-\tau||v_n(s)|\,ds\Big|+T\Big|\int_\tau^t |v_n(s)|\,ds\Big|.\end{aligned} \tag{6}$$

Since $v_n \in L^1(J,\mathbb{R})$, the right hand side of (6) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F^1)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S_F^1)(x)$ is a compact set in X. Therefore, Q is a compact-valued multi-valued operator on $[a,b]$.

Step II : Secondly we show that Q is right monotone increasing and maps $[a,b]$ into itself. Let $x, y \in [a,b]$ be such that $x \leq y$. Since $x \mapsto F(t,x)$ is right monotone increasing, one has $F(t,x) \overset{i}{\leq} F(t,y)$. As a result, we have from hypothesis (H_2) that $S_F^1(x) \overset{i}{\leq} S_F^1(y)$. Hence $Q(x) \overset{i}{\leq} Q(y)$. From (H_3) it follows that $a \leq Qa$ and $Qb \leq b$. Now Q is right monotone increasing, so we have

$$a \leq Qa \overset{i}{\leq} Qx \overset{i}{\leq} Qb \leq b$$

for all $x \in [a,b]$. Hence Q defines a multi-valued operator $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$.

Step III : Finally, let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a sequence in $Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$. We shall show that $\{y_n\}$ has a cluster point. This is achieved by showing that $\{y_n\}$ is uniformly bounded and equi-continuous sequence.

Case I : First we show that $\{y_n\}$ is uniformly bounded sequence. By definition of $\{y_n\}$ there is a $v_n \in S_F^1(x_n)$ such that

$$y_n(t) = x_0 + x_1 t + \int_0^t (t-s) v_n(s)\, ds,\ t \in J.$$

Therefore

$$\begin{aligned}
|y_n(t)| &\leq |x_0| + |x_1 t| + \int_0^t |t-s||v_n(s)|\, ds \\
&\leq |x_0| + |x_1|T + T\int_0^t \|F(s, x_n(s))\|\, ds \\
&\leq |x_0| + |x_1|T + T\int_0^T h_r(s)\, ds \\
&\leq |x_0| + \big(|x_1| + \|h_r(s)\|_{L^1}\big)T
\end{aligned}$$

for all $t \in J$, where $r = \|a\| + \|b\|$. Taking supremum over t,

$$\|y_n\| \leq |x_0| + \big(|x_1| + \|h_r(s)\|_{L^1}\big)T$$

which shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a,b])$.

Next we show that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Let $t, \tau \in J$.

Then we have

$$\begin{aligned}
|y_n(t) - y_n(\tau)| &\le |tx_1 - \tau x_1| + \Big| \int_0^t (t-s) v_n(s)\,ds - \int_0^\tau (\tau - s) v_n(s)\,ds \Big| \\
&\le |x_1||t-\tau| + \Big| \int_0^t (t-s) v_n(s)\,ds - \int_0^t (\tau - s) v_n(s)\,ds \Big| \\
&\quad + \Big| \int_0^t (\tau-s) v_n(s)\,ds - \int_0^\tau (\tau - s) v_n(s)\,ds \Big| \\
&\le |x_1||t-\tau| + \Big| \int_0^t (t-\tau) v_n(s)\,ds \Big| + \Big| \int_\tau^t (\tau - s) v_n(s)\,ds \Big| \\
&\le |x_1|\,|t-\tau| + \Big| \int_0^T |t-\tau||v_n(s)|\,ds \Big| + \Big| \int_\tau^t |\tau - s||v_n(s)|\,ds \Big| \\
&\le |x_1|\,|t-\tau| + \int_0^T |t-\tau| h_r(s)\,ds + T\Big| \int_\tau^t h_r(s)\,ds \Big| \\
&\le \big(|x_1| + \|h_r\|_{L^1}\big)|t-\tau| + |p(t) - p(\tau)|,
\end{aligned}$$

where $p(t) = T \displaystyle\int_0^t h_r(s)\,ds$. From the above inequality it follows that

$$|y_n(t) - y_n(\tau)| \to 0 \quad \text{as } t \to \tau.$$

This shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Now $\{y_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point in view of Arzelà-Ascoli theorem. Now the desired conclusion follows by an application of Theorem 1.

To prove the next result, we need the following definitions.

DEFINITION 6. A multi-valued function $\beta : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called $L^1_{\mathbb{R}}$-Chandrabhan if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $\beta(t,x)$ is right monotone increasing in x almost everywhere for $t \in J$, and

(iii) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\|\beta(t,x)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t,x)\} \le h(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

REMARK 1. Note that if the multi-valued function $\beta(t,x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then it is measurable in t and integrably bounded on $J \times \mathbb{R}$, and so, by a selection theorem, S^1_β has non-empty values, that is,

$$S^1_\beta(x) = \{u \in L^1(J,\mathbb{R}) \mid u(t) \in \beta(t,x) \text{ a.e. } t \in J\} \ne \emptyset$$

for all $x \in \mathbb{R}$. See Deimling [3] and the references therein for the details.

We use the following hypotheses in the sequel.

(H_5) F is L^1_R-Chandrabhan.

(H_6) The map $x \mapsto S^1_F(x)$ is right monotone increasing in $x \in \mathbb{R}$.

THEOREM 3. Assume that the hypotheses (H_1) and (H_5)-(H_6) hold. Then the DI (1) has a solution on J.

PROOF. Obviously the hypotheses (H_2) and (H_3) of Theorem 3.1 hold in view of Remark 3.1. Define two function $a, b : J \to \mathbb{R}$ by

$$a(t) = x_0 + x_1 t - \int_0^t (t-s)h(s)\,ds,$$

and

$$b(t) = x_0 + x_1 t + \int_0^t (t-s)h(s)\,ds.$$

It is easy to verify that a and b are respectively the lower and upper bounds of the DI (1) on J with $a \leq b$. Thus (H_4) holds and now the desired conclusion follows by an application of Theorem 2.

4 An Example

Let $J = [0, 1]$ and define a multi-valued function $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ by

$$F(t,x) = \begin{cases} \{0\} & x < 0 \\ [0, p(t)[x]] & x \in [0,2] \\ \{3\} & x > 2 \end{cases} \tag{7}$$

for all $t \in J$, where $p \in L^1(J, \mathbb{R}^+)$ and $[x]$ is a greatest integer not greater than x. Now consider the DI

$$\left.\begin{array}{l} x''(t) \in F(t, x(t)) \text{ a.e. } t \in J \\ x(0) = 0,\ x'(0) = 1. \end{array}\right\} \tag{8}$$

Clearly the multi-valued function F satisfies all the hypotheses of Theorem 3 with $a(t) = t$ and $b(t) = 2\int_0^t (t-s)p(s)\,ds + \frac{3}{2}t^2 + t$ for $t \in J$. Hence the DI (8) has a solution on J.

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